Superconductivity in disordered materials (*)

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Summary. — The real-space single-particle Green's function formalism is extended to the attractive disordered Hubbard Hamiltonian by matrix Nambu notation, thus providing a general approach to the superconductive instability of a disordered system in quantum mean-field approximation. Configurational averages may be performed by a recent method, already employed for the repulsive model, thus allowing for spatial fluctuations of the local order parameter. The method is shown to be a natural way of investigating the validity limit of the Anderson theorem. An exact analytical solution is provided for a Bethe lattice with no diagonal disorder and infinitely large connectivity.

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1. – Introduction

A full description of the interplay between disorder and interaction is far from being achieved, and the challenging problem of a many-body interacting electronic system in a random potential is still attracting considerable interest [1]. In the presence of a relevant repulsive interaction such problem plays an important role for the study of the metal-insulator transition [1, 2]. Conversely, a relevant attractive interaction gives rise to a superconductive instability which enriches the phase diagram of a generic disordered material [3]. Moreover, most strong-coupling mechanisms for high-$T_c$ superconductivity [4] are based on a direct-space short-range interaction giving rise to a novel kind of superconductive correlation on a short length scale, surviving some degree of disorder. Within Mean Field (MF) the superconductive phase had been shown [5] to persist up to the extreme on-site localization, but the space fluctuations of the order parameter were expected to yield a degradation of supercon-
ductivity [6]. In fact, increasing the degree of disorder, a superconductor-insulator transition is observed [3], even at $T = 0$, in contrast to the Anderson theorem [7]. On this subject, recent theoretical work has been devoted to the dirty boson approach from the analytical [8] and numerical [9] point of view.

The random Hubbard Hamiltonian is the simplest fermionic model which is still capable of describing the essential aspects of the problem. Such Hamiltonian has been intensively studied for the repulsive interaction, and a statistical mean-field approach has recently been employed in order to describe the spatial fluctuations of charge and local moments in the presence of strong diagonal disorder [2, 10]. Some numerical work has been reported [11] for the attractive interaction, while analytical diagrammatic theories for the Density of States [12] and for the conductivity [13] have been developed in the non-interacting or Hartree-Fock (HF) limit, making use of the direct-space Green's function formalism.

In this paper such formalism is generalized to the attractive interaction at MF level, in order to provide a general analytical scheme for the superconductive instability in the presence of strong diagonal and off-diagonal disorder. We show how most of the diagrammatic theories developed for the single-particle Density of States (DoS) can be easily extended to the attractive Hamiltonian in the MF approximation by adopting the standard Nambu [14] matrix notation.

As an example we provide the exact MF solution for a Bethe lattice with infinite connectivity and no diagonal disorder at $T = 0$. The extension to finite temperature and to a generic distribution for the diagonal disorder is straightforward but requires some numerical work. Such formalism could be employed for a full study of the spatial fluctuations of the local order parameter, in analogy to the statistical MF approach for the Hartree-Fock repulsive interaction [2, 10], thus exploring the reliability of the Anderson theorem [7].

This paper is organized as follows: sect. 2 illustrates the method, while in sect. 3 an analytical solution is provided for the special case of a Bethe lattice; a discussion on possible extensions of the method follows in sect. 4, and some details regarding sect. 3 are reported in appendix A.

2. – Method

The disordered Hubbard Hamiltonian is defined as

$$\tilde{H} = \sum_{ij, \sigma} t_{ij} a_{ij \sigma}^\dagger a_{ij \sigma} + \sum_{i\sigma} (\epsilon_i - \mu) a_{i\sigma}^\dagger a_{i\sigma} + U \sum_i \hat{n}_i \uparrow \cdot \hat{n}_i \downarrow ,$$

where $a_{i\sigma}$ annihilates an electron in the Wannier state centered on site $i$ of a random lattice, with spin projection $\sigma = \{ \uparrow, \downarrow \}$; $\hat{n}_{i\sigma} = a_{i\sigma}^\dagger a_{i\sigma}$; $\epsilon_i$ is a random variable with a given distribution function $p(\epsilon)$, and $\mu$ is a fixed chemical potential determining the average density $n = \langle n_\uparrow \rangle + \langle n_\downarrow \rangle$, where $n_{i\sigma} = \langle \hat{n}_{i\sigma} \rangle$ (quantum average) and $n = \langle n_{i\sigma} \rangle = \langle \hat{n}_{i\sigma} \rangle$ (configurational average). The hopping terms $t_{ij}$ may be considered as a given function of the random distance between sites $i$ and $j$ on the generic lattice. Then the sums over $\{ ij \}$ run over the $N$ sites of such lattice.

As usual, the effective MF Hamiltonian follows by neglecting second-order terms with respect to fluctuations in the interaction. Discarding some constant terms

$$\tilde{H}_{\text{MF}} = \sum_{ij, \sigma} t_{ij} a_{ij \sigma}^\dagger a_{ij \sigma} + \sum_{i\sigma} (\epsilon_i - \mu) a_{i\sigma}^\dagger a_{i\sigma} + \sum_i \Delta_i a_{i \uparrow \dagger} a_{i \downarrow} + \sum_i \Delta_i a_{i \downarrow \dagger} a_{i \uparrow} ,$$
where the local order parameter
\[ \Delta_i = U(a_i \downarrow a_i \uparrow) \]
contains a quantum average but is not averaged over the configurations, thus allowing for spatial fluctuations. The diagonal term \( \epsilon_i \) is the self-consistent unrestricted HF spin-dependent local level
\[ \epsilon_i \downarrow = \epsilon_i + Un_i \downarrow. \]
Both \( \Delta_i \) and \( n_i \) are local self-consistent fields determined through their definitions. This is the standard quantum MF approximation which must be distinguished from the statistical MF dealing with configurational averages.

In the following discussion we restrict ourselves to the \( T = 0 \) case since the extension to finite temperature is straightforward. The single-particle Green’s function is defined according to
\[ G_{ij}^\sigma(t, t') = -i \langle T[a_{i\sigma}(t), a_{j\sigma}^\dagger(t')] \rangle. \]
We introduce the anomalous functions
\[ F_{ij}(t, t') = -i \langle T[a_i \uparrow (t), a_j \downarrow (t')] \rangle, \]
\[ F_{ij}^\dagger(t, t') = -i \langle T[a_i \downarrow (t), a_j \uparrow (t')] \rangle, \]
then, in terms of the Fourier transforms, the equation of motion yields
\[ (\omega - \epsilon_i \uparrow + \mu) G_{ij}^\uparrow (\omega) = \delta_{ij} + \sum_k t_{ik} G_{kj}^\dagger (\omega) + \Delta_i F_{ij}^\dagger (\omega), \]
\[ (-\omega - \epsilon_i \downarrow + \mu) G_{ij}^\dagger (\omega) = \delta_{ij} + \sum_k t_{ik}^\times G_{kj}^\dagger (\omega) + \Delta_i^\times F_{ij} (\omega), \]
\[ (\omega - \epsilon_i \uparrow + \mu) F_{ij} (\omega) = \sum_k t_{ik} F_{kj} (\omega) - \Delta_i G_{ij}^\dagger (\omega), \]
\[ (-\omega - \epsilon_i \downarrow + \mu) F_{ij}^\dagger (\omega) = \sum_k t_{ik}^\times F_{kj}^\dagger (\omega) - \Delta_i^\times G_{ij} (\omega). \]
Analogous equations hold for the retarded and advanced functions, and as usual the imaginary part of \( \omega \) is determined by the boundary conditions.

Adopting the definitions
\[ H_{ik}^\sigma(\omega) = (\omega - \epsilon_i \sigma + \mu) \delta_{ik} - t_{ik}, \]
\[ \tilde{H}_{ik} = \begin{pmatrix} H_{ik}^\uparrow (\omega) & \delta_{ik} \Delta_i \\ -\delta_{ik} \Delta_i^\times & H_{ki}^\dagger (-\omega) \end{pmatrix}; \quad \tilde{G}_{kj} = \begin{pmatrix} G_{kj}^\uparrow (\omega) & F_{kj} (\omega) \\ -F_{kj}^\dagger (\omega) & G_{kj}^\dagger (-\omega) \end{pmatrix}, \]
the set of coupled equations (7) may be written in Nambu [14] matrix formalism
\[ \sum_k \tilde{H}_{ik} \cdot \tilde{G}_{kj} = \tilde{\delta}_{ij}, \]
where \( \tilde{\delta}_{ij} \) is the \( 2 \times 2 \) identity matrix \( \tilde{I} \) times \( \delta_{ij} \).

In order to make contact with the diagrammatic expansions developed for the single-particle Green’s function, we introduce a slightly different matrix formalism by
adopting the following definitions:

\[
\tilde{\Omega}_i = \begin{pmatrix}
(\omega - \varepsilon_{i \uparrow} + \mu) & \Delta_i \\
-\Delta_i^* & (-\omega - \varepsilon_{i \downarrow} + \mu)
\end{pmatrix}; \quad \tilde{V}_{ik} = \begin{pmatrix} t_{ik} & 0 \\ 0 & t_{i\downarrow}
\end{pmatrix}.
\]

Equation (10) then reads

\[
\tilde{\Omega}_i \tilde{G}_{ij} = \delta_{ij} + \sum_k \tilde{V}_{ik} \tilde{G}_{kj}.
\]

We observe that for \( \Delta_\downarrow = 0 \) the problem separates and we recover the standard equation of motion for the single-particle Green’s function. However, even for \( \Delta_\downarrow \neq 0 \), the matrix \( \tilde{V} \) is diagonal and, by iteration, eq. (12) gives rise to a locator expansion which is formally equivalent to that obtained for the single-component Green’s function. Then any of the approximate diagrammatic theories already developed from the locator expansion can be easily extended to the matrix formalism.

Self-consistency requires that, according to eq. (3)

\[
\Delta_i = -i U F_{\uparrow\downarrow}(t^+, t) = -i U \int \frac{d\omega}{2\pi} F_{\uparrow\downarrow}(\omega) e^{-i\omega t^+},
\]

which reduces to a sum over the spectral weights, or in terms of retarded and advanced functions

\[
\Delta_i = \frac{U}{2\pi i} \int_0^\infty \left[ F_{\uparrow\downarrow}^R(\omega) - F_{\uparrow\downarrow}^A(\omega) \right] d\omega,
\]

while the chemical potential \( \mu \) is defined by

\[
n = -\frac{1}{\pi} \sum_{\varepsilon_i} \int_{-\infty}^0 \text{Im} \, G_{\uparrow\downarrow}^{sR}(\omega) \, d\omega.
\]

The BCS limit is recovered neglecting the disorder: for a regular lattice without diagonal disorder \( \varepsilon_i = 0 \), assuming \( \Delta_i = \Delta \), in the reciprocal space, eq. (12) reads

\[
\tilde{\Omega} \tilde{G}(k) = \tilde{T} + t(k) \tilde{G}(k),
\]

where \( t(k) \), the Fourier transform of \( t_{ij} \), is the zero-order band energy. Equation (16) may be solved yielding

\[
\tilde{G}(k) = \left[ \tilde{\Omega} - t(k) \right]^{-1}
\]

and for the anomalous component

\[
F_{\uparrow\downarrow}^R (k, \omega) = \frac{\Delta}{2E(k)} \left( \frac{1}{\omega - E(k) \pm i\delta^+} - \frac{1}{\omega + E(k) \pm i\delta^+} \right),
\]

where \( E(k) \) is the band energy of the single-particle Green’s function.
where \( E(k) = \sqrt{(t(k) - \mu)^2 + |\Delta|^2} \). Finally, inserting in (14), we recover the BCS gap equation

\[
\Delta = -\frac{U}{2N} \sum_k \frac{\Delta}{E(k)}.
\]

For a generic topological and diagonal disorder, inverting the matrix \( \Omega_i \), equation (12) may be iterated as

\[
\bar{G}_{ki} = \frac{1}{\Omega_{ki}} \bar{V}_{ki} \frac{1}{\Omega_{ii}} + \frac{1}{\Omega_{ki}} \sum_l \bar{V}_{kl} \frac{1}{\Omega_{li}} \bar{V}_{li} \frac{1}{\Omega_{ii}} + \ldots,
\]
where \( k \neq i \) (off-diagonal), while the diagonal function follows directly from eq. (12):

\[
\bar{\Omega}_i \bar{G}_{ii} = 1 + \sum_k \bar{V}_{ki} \bar{G}_{ki}.
\]

Inserting eq. (20) in eq. (21), it is evident that the product \( \bar{\Omega}_i \bar{G}_{ii} \) gives the sum of all the chain diagrams which start and end up on the same site \( i \). Then, replacing the matrix \( \bar{\Omega}_i^{-1} \) by \( \bar{G}_{ii} \) at each stage in the locator expansion (20), we obtain a renormalized expansion free of one-articulated points

\[
\bar{G}_{ki} = \bar{G}_{ik} \bar{V}_{ki} \bar{G}_{ii} + \sum_{kl} \bar{G}_{kk} \bar{V}_{kl} \bar{G}_{li} \bar{G}_{ii} + \ldots.
\]

Here and in the following discussion some careful restriction on the allowed lattice sites is required in order to avoid double counting of terms in the renormalized expansion. The analysis is equivalent to the standard diagrammatic theory for the single-component Green’s function, thus we omit the details [15]. That said, in terms of the local self-energy matrix

\[
\bar{S}_i = \sum_{k \neq i} \bar{V}_{ik} \bar{G}_{kk} \bar{V}_{ki} + \sum_{kl \neq i} \bar{V}_{ik} \bar{G}_{kk} \bar{V}_{kl} \bar{G}_{li} \bar{G}_{ii} + \ldots
\]

eq. (21) reads

\[
\bar{\Omega}_i \bar{G}_{ii} = 1 + \bar{S}_i \bar{G}_{ii},
\]

which can be inverted as

\[
\bar{G}_{ii} = [\bar{\Omega}_i - \bar{S}_i]^{-1}.
\]

Any approximate sum of the expansion (23) plays the role of a closure relation for eq. (25), which can be solved for any finite ensemble of sites.

A very simple approximation, known as Hubbard approximation [12, 16], consists in retaining only the first term in the expansion (23) for the self-energy

\[
\bar{S}_i \approx \sum_k |t_{ik}|^2 \bar{G}_{kk}.
\]

Such relation becomes exact on a Bethe lattice where the only closed loop, free of one-articulated points, is just the two-site term. For a generic random lattice such approximate closure relation has often been regarded as the first step towards a more realistic treatment of disordered systems [17].
A full solution of the problem for a macroscopic ensemble of $N$ sites requires a configuration average of the Green’s function that we have avoided up to now. In order to incorporate some degree of spatial fluctuation for the local fields, a two-step averaging has recently been proposed [10]. A generalization of the method follows, according to the essential idea that the spatial fluctuation of any local field is strongly correlated to the fluctuation of the local energy level $\varepsilon_i$. In general we may undertake the following steps: i) perform a partial average over a restricted ensemble only containing lattice sites with the same energy level $\varepsilon_i = \varepsilon$; ii) then perform an average over the level $\varepsilon$ according to a given probability distribution $p(\varepsilon)$.

Finally we may neglect fluctuations for the step i), thus replacing any local quantity $\phi_i$ by its partial average $\phi(\varepsilon)$. A key point is that two sites with the same energy level are unlikely to be close in space, then the neglected fluctuations mainly regard lattice sites which are far from each other.

According to such program we may replace the matrix $\tilde{G}_{ii}$ by its partial average $\tilde{G}(\varepsilon)$, and evaluate the fully averaged Green’s functions as

$$\tilde{G} = \int_{-\infty}^{+\infty} P(\varepsilon) \tilde{G}(\varepsilon) \, d\varepsilon. \quad (27)$$

Any approximate evaluation of the expansion (23) gives rise to a closure relation, relating the self-energy matrix $\tilde{S}$ to the set of $N \{\tilde{G}_{ii}\}$ matrices. According to our prescription such closure relation yields the partial average $\tilde{S}(\varepsilon)$ as a functional of the partial average $\tilde{G}(\varepsilon')$:

$$\tilde{S}(\varepsilon) \equiv \tilde{S}[\varepsilon, \tilde{G}(\varepsilon')]. \quad (28)$$

For instance, the simple approximate closure relation (26) gives rise to a simple functional which does not depend explicitly on the energy level

$$\tilde{S}(\varepsilon) \equiv \tilde{S}[\tilde{G}] = J \int_{-\infty}^{+\infty} P(\varepsilon') \tilde{G}(\varepsilon') \, d\varepsilon', \quad (29)$$

where we have assumed the statistical independence between diagonal and spatial disorder and

$$J = z \int d^3 R g_\| (R) |t(R)|^2. \quad (30)$$

with $z$ being the average coordination number and $g_\| (R)$ the pair distribution function. We observe that such closure relation becomes exact for a Bethe lattice with infinitely large connectivity.

Now eq. (25) reads

$$\tilde{G}(\varepsilon) = [\tilde{\Omega}(\varepsilon) - \tilde{S}]^{-1} \quad (31)$$

and can be solved together with eq. (29). Full self-consistency requires an iterative procedure with the field $\Delta(\varepsilon)$ determined through eq. (14) at each step.

3. – Bethe lattice

In the previous section we discussed the general extension to matrix formalism of diagrammatic theories for the single-particle Green’s function of a disordered system. Here we focus on a specific model which can be exactly solved: namely a Bethe lattice
with nearest-neighbour hopping, no diagonal disorder and infinitely large connectivity.
In the presence of diagonal disorder a numerical solution can be achieved through eqs. (29) and (31). The model may be thought as approximating a real random lattice, or by itself as an example of exactly solvable problem.

Without diagonal disorder \((\varepsilon = 0)\) the probability distribution \(p(\varepsilon)\) reduces to a \(\delta\) distribution and eqs. (29), (31) yield

\[
(\bar{\Omega} - \bar{S}) \bar{S} = \bar{S}(\bar{\Omega} - \bar{S}) = J\hat{1}.
\]

Here the matrix \(\bar{\Omega}\) is obtained from the definition (11) of \(\bar{\Omega}\), replacing \(\Delta\) by \(\Delta\) and setting \(\varepsilon\) to zero (the HF shift is constant and can be ignored).

It is evident from eq. (32) that \(\bar{S}\) and \(\bar{\Omega}\) must be commuting matrices. Equation (32) is then equivalent to

\[
\left(\bar{S} - \frac{\bar{\Omega}}{2}\right)^2 = \frac{\bar{\Omega}^2}{4} - J\hat{1}.
\]

Denoting by \(\sqrt{A}\) the square root of a matrix, defined as \((\sqrt{A})^2 = A\), we may formally solve the problem in terms of the self-energy matrix

\[
\bar{S} = \frac{\bar{\Omega}}{2} \pm \sqrt{\frac{\bar{\Omega}^2}{4} - J\hat{1}}.
\]

The square root can be exactly evaluated as shown in appendix A: the resulting self-energy matrix is

\[
\bar{S} = \frac{\bar{\Omega}}{2} + \frac{1}{2} \begin{pmatrix}
\sqrt{B_+} & \frac{2\mu\Delta}{\sqrt{B_+} + \sqrt{B_-}} \\
-\frac{2\mu\Delta^*}{\sqrt{B_+} + \sqrt{B_-}} & \sqrt{B_-}
\end{pmatrix},
\]

where

\[
\sqrt{B_\pm} = \sqrt{b_\pm + \frac{4\mu^2 |\Delta|^2}{b_+ b_- + 2 \sqrt{b_+ b_- + 4\mu^2 |\Delta|^2}}}
\]

and

\[
b_\pm = (\pm \omega + \mu)^2 - |\Delta|^2 - 4J.
\]

In eqs. (35) and (36) each square root incorporates an independent \(\pm\) sign. Each of them must be fixed by imposing the correct boundary conditions. For instance from eq. (29) the averaged single-particle Green's functions are \(\bar{G} = (1/J) \bar{S}\). The diagonal elements are

\[
G^\uparrow (\omega) = \frac{\omega + \mu}{2J} + \frac{1}{2J} \sqrt{B_+},
\]

\[
G^\downarrow (-\omega) = \frac{-\omega + \mu}{2J} + \frac{1}{2J} \sqrt{B_-},
\]
and the single-particle DoS follows

\[
D_\alpha(\pm \omega) = \mp \frac{1}{2\pi J} \text{Im} \sqrt{B_\pm(\omega + i\theta^+)} > 0.
\]

The off-diagonal elements yield

\[
F_R(\omega) - F_A(\omega) = \frac{\Delta}{4\omega J} \left[ 2 \text{Im} \sqrt{B_+(\omega + i\theta^+)} + 2 \text{Im} \sqrt{B_-(\omega + i\theta^+)} \right]
\]

or in terms of the DoS

\[
F_R(\omega) - F_A(\omega) = -\frac{i\pi \Delta}{\omega} \left[ D_+(\omega) + D_-(\omega) \right].
\]

Finally, self-consistency requires from eq. (14)

\[
\frac{1}{U} = -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{D^\alpha(\omega)}{|\omega|} d\omega,
\]

which is the usual gap equation.

In units of the zero-order \((U = 0)\) half band-width \(W = 2\sqrt{J}\), the DoS is reported in fig. 1 for \(U = -0.5\) and \(n_\uparrow = n_\downarrow = 0.2\). The chemical potential is \(\mu = -0.5\) and the gap \(\Delta = 0.04\). For comparison the zero-order semi-elliptic DoS is reported. As shown in fig. 2, for larger values of the gap, the DoS acquires some extra jumps in the first derivative as a consequence of the second square root root in eq. (36). Moreover, a long tail

Fig. 1. – The density of states for \(U = -0.5\) and \(n_\uparrow = 0.2\). The gap parameter is \(\Delta = 0.04\) and the Fermi level \(\mu = -0.5\). All energies are in units of the zero-order half band-width \(W = 2\sqrt{J}\). The corresponding zero-order semi-elliptic density of states is reported as a dashed line.
Fig. 2. – As fig. 1 for $U = -1.0$ and $n_s = 0.22$. The gap parameter is $\Delta = 0.26$ and the Fermi level $\mu = -0.5$.

Fig. 3. – The gap parameter $\Delta$ vs. the Fermi level $\mu$, for $U = -1.0$. Units are adimensional as in fig. 1.

shows up at the band bottom, thus incorporating part of the weight lost at the Fermi energy. The physical meaning of such features is not very clear.

As a function of the chemical potential $\mu$, and for a fixed value of $U$, the trend of the gap parameter is quite regular and reflects the behaviour of the DoS, as shown in fig. 3 for $U = -1.0$. 
4. – Discussion

In the previous section we discussed the exact analytical solution for a simple approximate model. As expected the gap parameter increases with increasing the strength $U$ of the attractive interaction. Thus $U$ is an important parameter of the model, determining the structure of the DoS which is directly related to the experimental observation. Especially when $U$ is small, as is usually the case, such behaviour is not trivial and is related to the degree of internal correlation for each Cooper pair.

In order to discuss the quantum transition from insulator to superconductor, the disordered Hubbard model has often been mapped on a hard-core boson Hamiltonian which is believed to retain the important physical aspects of the problem [8, 9]. That approach allows a simpler description of the critical aspects, but is based on the assumption that the Cooper pairs retain their strong correlation across the transition, as we would expect in the presence of a strong negative $U$. In fact, in the limit $|U| \rightarrow \infty$, we may forget about the internal degrees of freedom of the Cooper pairs, and regard each pair as a boson.

Such approach is questioned in the opposite limit $U \rightarrow 0$ since then the weakening of the pairs could play an important role in the transition. In the language of eq. (3) the local order parameter $\Delta_i$ is characterized by its modulus $|\Delta_i|$ and its phase $\arg \Delta_i$. Phase ordering is the central aspect which has been addressed by using bosonic Hamiltonians [8, 9] or related models [18-20]. However, this requires a non-zero modulus $|\Delta_i|$ which is not guaranteed by any small interaction energy $U$ in the presence of diagonal and topological disorder. The weakening of the modulus $|\Delta_i|$, induced by disorder and spatial fluctuations, can only be addressed by a study of the full Hubbard Hamiltonian, thus restoring the role played by the interaction energy $U$. We expect that at the transition point both the aspects could conspire, determining the eventual critical properties.

In order to clarify such point of view, we could compare such scenario with the corresponding metal-insulator transition occurring in the repulsive model. The role of correlations in the localization transition is not well understood, and especially the effect of the interaction on the critical properties. Moreover the interaction is strongly related to the local moment formation which shows up close to the metal-insulator transition. The presence of local moments affects the averaged DoS which, besides, is perfectly continuous at the transition. That does not mean that interactions do not play any role for determining the critical properties.

Back to the attractive model, we expect the averaged DoS to be continuous at a transition determined by phase ordering. But we must take in due account the role played by the internal correlation of the pairs. As for local moments, the modulus $|\Delta_i|$ is strongly related to the attractive interaction $U$, and affects the DoS. The weakening of $|\Delta_i|$, even if proven to be continuous at the transition, is expected to play an important role.

On the other hand, the full disordered Hubbard model is not very suited for analytical solutions. Some progress has recently come on the numerical side, by use of quantum Monte Carlo simulations. For small $U$ values, the MF approach should be reliable, thus mapping the problem on a single-particle Anderson model as is evident from eq. (12).

The exact analytical solution provided in sect. 3 could be of interest by itself as an example of solvable problem, but its importance must be related to its eventual
extension to diagonal disorder. In fact, such solution does not address the problem of verifying the validity of the Anderson theorem: the gap parameter is fixed at its average value and spatial fluctuations are neglected. Moreover such solution is exact in the infinite dimensional limit, and no quantum phase transition may occur in such limit. In fact for any $U < 0$ eq. (42) always predicts the presence of a superconductive phase. Then, at the present stage, no serious comparison may be considered between such simple solution and the critical properties of the boson models, which have been generally studied on a low-dimensional lattice.

Allowing for spatial fluctuation, as discussed in sect. 2, would probably give us new insight into the problem: the quantum MF approximation does not imply a statistical MF, and some aspects regarding the spatial fluctuations of the order parameter can be extracted by a careful handling of the configurational average. The analogy of eq. (20) with the corresponding locator expansion for the Anderson Hamiltonian could be employed in order to adopt approximations that have shown to be successful for the localization problem. For instance eq. (26), which is exact on a Bethe lattice, could be the starting point for an analysis based on distribution functions for the order parameter. Notice that eqs. (14), (25) and (26) are not averaged, and thus contain information about phase ordering.

The same scheme can be extended to more realistic lattice models, and to diagonal disorder which probably plays a major role in the degradation process of superconductivity. A self-consistent calculation starting from eqs. (29) and (31) would be the analogous of the recent work on the repulsive Hubbard model [2, 10] which has been found successful for describing local moment formation in liquid alloys [10] and in compensated semiconductors [2]. The outcome would be a qualitative discussion of the weakening of $|\Delta|$ as a consequence of the interplay between disorder and interaction. Of course even more refined theories for the DoS [12] could be employed using the analogy of the matrix equation (12) with respect to the standard equation of motion for the single-component Green’s function.

### Appendix A

We want to evaluate the square root of a matrix $\hat{A}$ defined, according to eq. (34), as

$$\hat{A} = \frac{\hat{b}}{4} + \frac{\mu}{2} \hat{d}, \quad (A.1)$$

where

$$\hat{b} = \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}; \quad \hat{d} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix}. \quad (A.2)$$

Since $\hat{b}$ is diagonal, its square root does not present any problem. As in the main text, in this appendix each square root incorporates a $\pm$ sign which must be fixed by physical arguments. Let us consider the matrix $\sqrt{\hat{b}/2 + (\mu/(2 \sqrt{\hat{b}})) \hat{d}}$ which can be regarded as an expansion of $\sqrt{\hat{A}}$ truncated to order $\Delta$. The square of such matrix differs from $\hat{A}$ already at order $\Delta$ since $\hat{b}$ and $\hat{d}$ do not commute. We may improve such ansatz by
noticing that, denoting \( z = \sqrt{b_+} + \sqrt{b_-} \),

\[
\begin{pmatrix}
\frac{1}{2} \sqrt{b_+} & \frac{\mu \Delta}{z} \\
-\frac{\mu \Delta^*}{z} & \frac{1}{2} \sqrt{b_-}
\end{pmatrix}^2 = \begin{pmatrix}
\frac{1}{4} b_+ - \frac{\mu^2 |\Delta|^2}{z^2} & \frac{\mu \Delta}{2} \\
-\frac{\mu \Delta^*}{2} & \frac{1}{4} b_- - \frac{\mu^2 |\Delta|^2}{z^2}
\end{pmatrix} = \bar{A} + O(\Delta^2).
\]  

Such approximate solution to order \( \Delta^2 \) yields a linearized gap equation.

The exact solution can be found by substituting new quantities \( B_\pm \) for \( b_\pm \) according to the definition

\[
B_\pm = b_\pm + 4 \mu^2 |\Delta|^2 / z^2
\]

since this way the diagonal terms in the squared matrix of eq. (A.3) reduce to \( (1 / 4) b_\pm \) as required. Of course, the variable \( z \) must be self-consistently determined as

\[
z = \sqrt{B_+} + \sqrt{B_-},
\]

then inserting eq. (A.4) in eq. (A.5) we find

\[
z^2 = b_+ + b_- + 2 \sqrt{b_+} b_- + 4 \mu^2 |\Delta|^2
\]

which in turn, inserted in eq. (A.4), gives the variables \( B_\pm \) and through eq. (A.3) the solution reported in eq. (35) of sect. 3.

REFERENCES