On averaging the Kubo-Hall conductivity of magnetic Bloch bands leading to Chern numbers (*)

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Summary. — We re-examine the topological approach to the integer quantum Hall effect in its original form where an average of the Kubo-Hall conductivity of a magnetic Bloch band has been considered (Phys. Rev. Lett., 49 (1982) 405). For the precise definition of this average it is crucial to make a sharp distinction between the discrete Bloch wave numbers $k_1$, $k_2$ and the two continuous integration parameters $\alpha_1$, $\alpha_2$. The average over the parameter domain $0 \leq \alpha_j < 2\pi$, $j = 1$, 2 of the Hall conductivity of the band is a sum of integrals, each associated with different discrete $k_1$, $k_2$. We show how this can be transformed into a single integral over the continuous magnetic Brillouin zone $0 \leq \alpha_j < 2\pi n_j$, $j = 1$, 2, $n_j = $ number of unit cells in $j$-direction, keeping $k_1$, $k_2$ fixed (which is the Chern number of the band times $\frac{e^2}{h}$). This average prescription for the Hall conductivity of a magnetic Bloch band is exactly the same as the one used for a many-body system in the presence of disorder (Phys. Rev. B, 31 (1985) 3372).

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1. Introduction

The topological approach to the integer quantum Hall effect (IQHE) goes back to the seminal paper by Thouless, Kohmoto, Nightingale, and den Nijs (TKNN) [1], who considered non-interacting electrons in a periodic substrate potential and expressed the average of the Kubo formula for the Hall conductivity in a form which was subsequently [2] interpreted as $\frac{e^2}{h}$ times a Chern number (the first Chern class of a principal fiber bundle on a torus).

The derivation in [1] (a more detailed presentation is given in [3]) of the result that the averaged Kubo-Hall conductivity of magnetic Bloch bands is equal to $\frac{e^2}{h}$ times

(*) The author of this paper has agreed to not receive the proofs for correction.
an integer, raises some mathematical questions, which are not addressed in the original papers.

The first point is the following. In [1, 3] the averaged Kubo formula of a fully occupied magnetic Bloch band is written as a single integral over the continuous Brillouin zone, by generalizing the discrete wave numbers \( k_1, k_2 \) to continuous variables. On the other hand, starting from the definition of the Kubo formula, its average is a priori a sum of integrals, each of which is associated with a different discrete \( k \)-vector \( (k_1, k_2) \). In this paper we will clarify some mathematical aspects and add the missing steps which are necessary to fully understand the averaging process for magnetic Bloch bands (keeping the same mathematical language in which the original papers [1, 4] have been written). In particular, we will show how the sum of integrals can be transformed into a single integral. We could not find this in the literature, at least not in the language of the original papers, which is accessible to a wider audience.

Secondly, in the framework of a many-body formulation including a disorder potential, the Kubo formula was averaged over the boundary conditions [4]. This raises the following question: are the averaging prescriptions of [1, 3] and of [4] exactly the same (same integration parameters, same integration domain), or only analogous or similar. From the text in [4] one might conclude that the latter is the case. On the other hand, physical consistency would require that the average prescriptions be identical, i.e. independent of the system considered.

The answer to these questions is also important in view of the abundant theoretical literature which relies on the original article [1] (see [5] for a recent review) and further, in view of the quite spectacular physical predictions which it implies (see the discussion at the end of sect. 3).

2. - Averaging the Kubo-Hall conductivity

The main task of the present paper will be to understand the precise mathematical definition of the integral for the averaged Kubo-Hall conductivity presented in [1] and further, to clarify its relation to the average performed in [4]. To this end it is helpful to first review the formulation of [4], where a many-particle system in the presence of disorder is considered. Here the geometry is a rectangular domain of dimension \( L_1 \times L_2 \), as in [1]. The essential point is that the boundary conditions are defined such that the wave functions change by a phase factor \( \exp\{i\alpha_j\} \), \( j = 1, 2 \), under the magnetic translation operators corresponding to an increase of \( x, y \) by \( L_1, L_2 \), respectively (see (2.4) of [4]; our notation \( (\alpha_1, \alpha_2) \) represents \( (\alpha L_1, \beta L_2) \) of [4]). The real numbers \( \alpha_j \) can always be written as \( \alpha_j = \alpha_j' + 2\pi m_j \), \( 0 \leq \alpha_j' < 2\pi \), with \( m_j \) integer. Each set \( (\alpha_1, \alpha_2) \), \( 0 \leq \alpha_1, \alpha_2 < 2\pi \), defines a different self-adjoint representation of the Hamiltonian \( H \) = \( H(\alpha_1, \alpha_2) \). The Hamiltonian \( H(\alpha_1, \alpha_2) \) depends on the parameters \( \alpha_1, \alpha_2 \) only through these boundary conditions.

By means of the unitary transformation \( U(\alpha_1, \alpha_2) \) (eq. (2.5) of [4], which is the many-particle form of the transformation (5) below) one obtains the Hamiltonian \( \tilde{H}(\alpha_1, \alpha_2) \), with eigenfunctions \( \tilde{\psi}(\alpha_1, \alpha_2) \), where the parameters \( \alpha_1 \) and \( \alpha_2 \) (times a constant factor) appear now as magnetic vector potentials in the two spatial directions, and \( \alpha_j \hbar c/e, j = 1, 2 \), can be considered as magnetic fluxes. Note that the self-adjoint representation of \( \tilde{H}(\alpha_1, \alpha_2) \) is fixed, i.e. independent of \( (\alpha_1, \alpha_2) \).
The Kubo formula for the (parameter dependent) Hall conductivity of the system considered in [4] can be cast into the form [4]

\[
\sigma(\alpha_1, \alpha_2) = (\text{i}e^2/\hbar)\left(\langle \partial \tilde{\psi}/\partial \alpha_1 | \partial \tilde{\psi}/\partial \alpha_2 \rangle - \langle \partial \tilde{\psi}/\partial \alpha_2 | \partial \tilde{\psi}/\partial \alpha_1 \rangle \right).
\]

The key point is now to replace \(\sigma(\alpha_1, \alpha_2)\) by its average over the two-dimensional parameter domain \(S = \{\alpha_1, \alpha_2 | 0 \leq \alpha_i \leq 2\pi, i = 1, 2\}\), see [4]. (i.e. in [4] the average is taken over all the non-equivalent self-adjoint representations of the original Hamiltonian \(H\)). This average can be written as

\[
\langle \sigma \rangle = \frac{1}{\pi} \int_S \sigma dS.
\]

Here \(\mathbf{V} = i(\nabla_a \tilde{\psi} \times |\nabla_a \tilde{\psi}|)\) and \(\tilde{\psi}\) is the ground state of the many-particle Hamiltonian \(\tilde{H}(\alpha_1, \alpha_2)\).

An integral of type (2) only depends on the subspaces spanned by the functions \(\tilde{\psi}(\alpha_1, \alpha_2)\) [6,2b]. Further, it has been recognized [2] that such an integral is a topological invariant equal to \(e^2/\hbar\) times an integer (a Chern number), provided \(S\) is a closed surface without a boundary, a torus in our case. From this it follows immediately that (2) represents such a topological invariant, provided the one-dimensional subspace defined by the function \(\tilde{\psi}(\alpha_1, \alpha_2)\) is periodic with period \(2\pi\) in each of the two parameters \(\alpha_1\) and \(\alpha_2\). But this periodicity requirement is not fulfilled by the function \(\tilde{\psi}(\alpha_1, \alpha_2)\) of [4], see, e.g. [7] or eq. (11) below. It however turns out that it is sufficient that the function \(\tilde{\psi}(\alpha_1, \alpha_2) = U^{-1}(\alpha_1, \alpha_2) \tilde{\psi}(\alpha_1, \alpha_2)\), i.e. the corresponding eigenfunctions of the original Hamiltonian \(\tilde{H}(\alpha_1, \alpha_2)\), has this periodicity property, see the discussion in sect. 3 of the present paper.

In opposition to [4], TKNN considered a periodic substrate potential \(V(x, y)\) with spatial period \(a\) and \(b\) in \(x\) and \(y\)-direction, and a magnetic field \(B\) in \(z\)-direction with special values, such that the magnetic unit cell \([qa, b]\) (\(q\) integer) contains an integer number \(p\) of flux quanta \(hc/e\) (\(p\) and \(q\) are mutual primes). A more detailed version of [1] is given in [3]. The one-electron Hamiltonian is

\[
H = \left[1/(2m)\right]\left[\hbar^2/\left(\epsilon(c) A(r)\right)^2 + V(x, y)\right].
\]

Here \(\epsilon\) is the charge of the particle and \(\text{curl } A = (0, 0, B)\). The eigenfunctions of (3) are magnetic Bloch functions

\[
\psi(k_1, k_2)(x, y) = \exp[ik_1 x + ik_2 y] u(k_1, k_2)(x, y)
\]

with eigenvalues \(E(k_1, k_2)\) and obey the generalized Bloch property (eq. (2.11) of [3]). The generalized crystal momenta are the discrete values \(k_j = 2\pi\hbar/L_j, j = 1, 2, h\) integer. Each Landau level is split into \(p\) subbands (magnetic Bloch bands) of equal weight. In (4) the subband index is omitted. Each subband corresponds to values of \(k_1, k_2\) restricted to the (discrete) magnetic Brillouin zone (MBZ) \(0 \leq k_i < 2\pi/(qa), 0 \leq k_2 < 2\pi/b\), which corresponds to \(n_1 = L_1/(qa)\) values of \(k_1\), and \(n_2 = L_2/b\) values of \(k_2\).

We now proceed in complete analogy to [4]. We first define \(\psi(k_1, k_2)(\alpha_1, \alpha_2)\) as an eigenfunction of \(H(\alpha_1, \alpha_2)\), which is now the one-electron Hamiltonian (3), defined with boundary conditions which are the one-electron version of (2.4) of [4]. These
boundary conditions are characterized by the parameters \((\alpha_1, \alpha_2)\), see above. Next we define

\[
(5) \quad \widetilde{\psi}(k_1, k_2)(\alpha_1, \alpha_2) = U(\alpha_1, \alpha_2)\psi(k_1, k_2)(\alpha_1, \alpha_2) = \exp\left[ -i\alpha_1 x/L_1 - i\alpha_2 y/L_2 \right] \psi(k_1, k_2)(\alpha_1, \alpha_2),
\]

which is the single-particle form of the transformation (2.5) of [4]. The transformed Hamiltonian \(U(\alpha_1, \alpha_2) H(\alpha_1, \alpha_2) U^{-1}(\alpha_1, \alpha_2)\) is again denoted \(\tilde{H}(\alpha_1, \alpha_2)\). Clearly,

\[
\widetilde{\psi}(k_1, k_2)(\alpha_1 = 0, \alpha_2 = 0) = \psi(k_1, k_2)(\alpha_1 = 0, \alpha_2 = 0) = \psi(k_1, k_2),
\]

where \(\psi(k_1, k_2)\) is a true magnetic Bloch function as defined by (4).

The starting point of [1] and [3] is the conventional form of the zero-temperature Kubo formula \(\sigma_{12}^{\text{Kubo}}\) for the Hall conductivity of a fully occupied magnetic Bloch band (see, e.g., eq. (3.3) of [3]). Expressing the velocity operators \(\tilde{v}(\alpha_1, \alpha_2)\) as \((L_j/\hbar) \partial \tilde{H}(\alpha_1, \alpha_2) / \partial \alpha_j\), \(j = 1, 2\) and using some manipulations (see [3]), \(\sigma_{12}^{\text{Kubo}}\) can be written in a new form,

\[
(6) \quad \sigma_{12}^{\text{Kubo}} = \sigma(\alpha_1 = 0, \alpha_2 = 0),
\]

where

\[
(7) \quad \sigma(\alpha_1, \alpha_2) = \sum_{k_1, k_2} \sigma[k_1, k_2](\alpha_1, \alpha_2), \quad (k_1, k_2) \in \text{MBZ},
\]

\[
(8) \quad \sigma[k_1, k_2](\alpha_1, \alpha_2) = (i e^2 / h) \left[ \left( \partial \tilde{\psi}[k_1, k_2] / \partial \alpha_1 \right) \left( \partial \tilde{\psi}[k_1, k_2] / \partial \alpha_2 \right) - \left( \partial \tilde{\psi}[k_1, k_2] / \partial \alpha_2 \right) \left( \partial \tilde{\psi}[k_1, k_2] / \partial \alpha_1 \right) \right]
\]

((6) and (7) are generalized notations which we will use in the following).

Expression (7) is a discrete sum running over the \(n_1 n_2\) wave vectors \((k_1, k_2)\) of the discrete MBZ. An integral can be obtained by averaging (7) over the continuous parameters \((\alpha_1, \alpha_2)\). For physical consistency we average exactly as in [4], i.e., over the domain \(0 \leq \alpha_j \leq 2\pi\), \(j = 1, 2\), which corresponds to an average over all non-equivalent self-adjoint representations of \(H(\alpha_1, \alpha_2)\), see the step from (1) to (2). This average leads to

\[
(9) \quad \sigma_{av} = \sum_{k_1, k_2} \sigma_{av}[k_1, k_2], \quad k_1, k_2 \in \text{MBZ}.
\]

Here \(\sigma_{av}[k_1, k_2]\) is defined as in (1) and (2), but where \(\tilde{\psi}\) represents now the one-electron function \(\tilde{\psi}[k_1, k_2](\alpha_1, \alpha_2)\). In (9) \(\sigma_{av}\) is still a sum of \(n_1 n_2\) integrals over the domain \(0 \leq \alpha_j \leq 2\pi\), \(j = 1, 2\). In the following we will show that this sum can be expressed as a single integral over the continuous domain \(0 \leq \alpha_j \leq 2\pi n_j\), \(j = 1, 2\) (which corresponds to the continuous MBZ in parameter space).

For the further developments the following two relations are important:

\[
(10) \quad U^{-1}(2\pi p_1, 2\pi p_2) \tilde{\psi}[k_1, k_2](\alpha_1 + 2\pi p_1, \alpha_2 + 2\pi p_2) = \tilde{\psi}[k_1 + 2\pi p_1/L_1, k_2 + 2\pi p_2/L_2](\alpha_1, \alpha_2) \exp[i f(\alpha_1, \alpha_2; p_1, p_2)], \quad 0 \leq \alpha_j \leq 2\pi n_j, \quad p_j \text{ integer}, \quad i = 1, 2,
\]
(11) \[ \tilde{\psi}(k, \alpha; 2\pi n_1 s, 2\pi n_2 t) = \]

\[ = U(2\pi n_1 s, 2\pi n_2 t) \tilde{\psi}(k, \alpha) \exp \{i(f(\alpha, \alpha; s, t)), \quad s, t \text{ integer}. \]

Here exp \{i(f(\alpha, \alpha; p_1, p_2))\} and exp \{i(f(\alpha, \alpha; s, t))\} are global phase factors. Equations (10) and (11) describe general properties of the eigenfunctions of parameter-dependent Hamiltonians of type \( \hat{H}(\alpha_1, \alpha_2) \) (which belong to a fixed self-adjoint representation, independent of \( \alpha_1, \alpha_2 \)). Such Hamiltonians have been investigated in [8] in a general and systematic way (\( \alpha_j \) represents \(-2\pi\theta(\alpha_j)\) of [8]). All physical properties of a state \( \tilde{\psi}(k_1, k_2)(\alpha_1, \alpha_2) \) are periodic in \( \alpha_j \) with period \( 2\pi n_j \). \( n_j \) is the number of unit cells in \( j \)-direction. This is just the smallest integer, such that the eigenvalues of \( \hat{H}(\alpha_1, \alpha_2) \) are periodic in \( \alpha_j \) with period \( 2\pi n_j \). Relation (10) will be derived in the appendix. Equation (11) has already been derived in [8]. Figure 1 illustrates relations (10) and (11).

We take now \( k_1, k_2 \) as fixed values and we define \( k_1' = k_1 + 2\pi p_1/L_1, \quad k_2' = k_2 + 2\pi p_2/L_2 \), where the integers \( p_1, p_2 \) vary such that \( (k_1', k_2') \) cover all the points in the discrete MBZ. Then (9) is calculated, where the sum is now taken over the \( k_1', k_2' \), which we have just defined. Further, in the definition (8) of \( \sigma(k_1', k_2')(\alpha_1, \alpha_2) \) we express \( \tilde{\psi}(k_1', k_2')(\alpha_1, \alpha_2) \) by its form on the left hand side of (10). In this way, using the fact that \( U^{-1}(2\pi p_1, 2\pi p_2) \) does not depend on \( \alpha_j \), one finds that \( \sigma_{av}(k_1', k_2') \) (which is the average of \( \sigma(k_1', k_2')(\alpha_1, \alpha_2) \) over \( 0 \leq \alpha_j < 2\pi \)) is equal to the average of \( \sigma(k_1, k_2) \)

\[ \text{Fig. 1. - The figure illustrates the parameter dependence of the eigenvalues } E(k_1, k_2)(\alpha_1, \alpha_2) \text{ of } \hat{H}(\alpha_1, \alpha_2), \text{ belonging to the eigenfunctions } \tilde{\psi}(k_1, k_2)(\alpha_1, \alpha_2). \text{ Only the variation with respect to one parameter, say } \alpha_1 = \alpha, \text{ is shown (} \alpha_2 \text{ is kept fixed). An example with } n_1 = 4 \text{ is shown, i.e. } k_1 = 2\pi n_1/L_1, \quad h = 0, 1, 2, 3. \quad E(k_1, k_2)(\alpha_1, \alpha_2) \equiv E(k_1)(\alpha) \equiv E_{h}(\alpha). \quad \text{Figure 1 illustrates that } E_{h}(\alpha) = E_{h+1}(\alpha - 2\pi t), \text{ where } E_{h}(\alpha) \equiv E_{h+1}(\alpha), \quad h, \text{ integer. The relation between the corresponding eigenfunctions } \tilde{\psi}_{h}(\alpha) \text{ is obtained from (10) and (11). E.g., } \tilde{\psi}_{0}(0 \leq \alpha < 2\pi) \text{ can be expressed on each subinterval } 2\pi h \leq \alpha < 2\pi(h + 1) \text{ as the transformation defined by (10) of the function } \tilde{\psi}_{0}(0 \leq \alpha < 2\pi), \quad h = 1, 2, 3. \text{ This is indicated by arrows for the case } h = 1, \text{ i.e. } \tilde{\psi}_{1}(0 \leq \alpha < 2\pi) \rightarrow \tilde{\psi}_{0}(2\pi \leq \alpha < 4\pi). \]
over the domain $2\pi p_j \leq \alpha_j \leq 2\pi(p_j + 1)$. Finally, summing over all $k_1', k_2'$ in (9) leads to

$$
\sigma_{\alpha\nu} = \left[\frac{1}{4\pi^2}\right] \int_{0}^{2\pi n_1} d\alpha_1 \int_{0}^{2\pi n_2} d\alpha_2 \sigma[k_1, k_2](\alpha_1, \alpha_2).
$$

Here $k_1, k_2$ is any fixed pair among the set of crystal momenta in the discrete MBZ, e.g., $k_1 = 0, k_2 = 0$. We thus have succeeded in expressing the sum of integrals over $0 \leq \alpha_j < 2\pi$ in (9) as a single double-integral over the continuous domain $0 \leq \alpha_j < 2\pi n_j$, $j = 1, 2$. Such a form is necessary for the topological argument of [1] and [3]. (For completeness we mention that, as a consequence of (10), the integral (12) is unchanged if the rectangular area of integration is shifted in parameter space by a constant vector.)

By construction, (12) is the Kubo-Hall conductivity of the magnetic Bloch band averaged over the domain $0 \leq \alpha_j \leq 2\pi$, $j = 1, 2$. As we have seen before, this restricted integration domain corresponds to the boundary condition average introduced in [4]. Our analysis shows, therefore, that the average prescriptions of [1, 3] (for a magnetic Bloch band) and of [4] (for a many-body system with disorder) are exactly the same. In both cases a Chern number is obtained by averaging the Kubo-Hall conductivity of the system over the same integration parameters and the same integration domain (corresponding to an average over all non-equivalent self-adjoint representations of the Hamiltonian $H$).

3. Discussion

The step from (9) to (12) is essential for the precise derivation of (12), i.e., for obtaining an integral over the continuous MBZ (which itself is a necessary prerequisite for the final result that the averaged Kubo-Hall conductivity of a magnetic Bloch band is equal to $(e^2/h)$ times an integer). However, this step from (9) to (12) is missing in [1] and [3]. In the development given in [1] and [3] only one set of parameters appears, which seems to be the formal amalgamation $k_j L_j + \alpha_j$ of the discrete $k_j$ with the continuous $\alpha_j$. However, as our analysis has shown, it is crucial to make a sharp distinction between these two parameters. (The difference in the two parameters can already be seen from the fact that the subspace defined by a function $\tilde{\psi}[k_1, k_2](\alpha_1, \alpha_2)$ is not periodic as a function of the continuous parameters $\alpha_1, \alpha_2$ (only the modulus of $\tilde{\psi}$ is periodic with period $2\pi n_j$, but not its phase function) but is periodic as a function of the discrete parameters $k_1$ and $k_2$ (when $k_j$ goes from one edge of the discrete MBZ to the opposite edge). The former follows from (11), the latter by putting (11) in (10)).

Finally we make some observations on how to obtain the value of (12). First, since the subspace defined by a single function $\tilde{\psi}[k_1, k_2](\alpha_1, \alpha_2)$ is not periodic in $\alpha_j$, the domain of integration in (12) is not equivalent to a torus. Therefore, at this stage, one cannot conclude that the value of (12) is equal to $e^2/h$ times a Chern number. However, it is possible to show that (12) is equal to another integral, which formally looks like (12) but which is defined by replacing $\tilde{\psi}[k_1, k_2](\alpha_1, \alpha_2)$ in (8) by $\psi[k_1, k_2](\alpha_1, \alpha_2)$. We denote this modified integral (12) by $\sigma_{a\nu}[\psi]$, and (12) itself by $\sigma_{a\nu}[\tilde{\psi}]$. From (5) and (11) it follows that the subspace defined by $\psi[k_1, k_2](\alpha_1, \alpha_2)$ is periodic in the MBZ with respect to the continuous variables $\alpha_1, \alpha_2$. Therefore (according to [2]) $\sigma_{a\nu}[\psi]$ is equal to a Chern number (times $e^2/h$).
It remains then to show that $\sigma_{av}[\psi]$ is equal to $\sigma_{av}[\bar{\psi}]$. To this end we assume that 

\[ \psi(k_1, k_2)[\alpha_1, \alpha_2] \]

can be chosen continuous as a function of $(\alpha_1, \alpha_2)$ in the entire MBZ. By means of Stokes' theorem the two surface integrals $\sigma_{av}[\psi]$ and $\sigma_{av}[\bar{\psi}]$ are then expressed as path integrals $\gamma[\psi; C]$ and $\gamma[\bar{\psi}; C]$ respectively, where the integration path $C$ is the contour of the MBZ in the space of the parameters $\alpha_1, \alpha_2$. These two path integrals have the mathematical form of Berry phase integrals [6]. From the general form of the Berry phase integral (eq. (6) of [6]), it is then straightforward to show by direct calculation that

\[ \gamma[\psi; C] = \gamma[\bar{\psi}; C] \] (using (5) and (11)). Hence $\sigma_{av}[\bar{\psi}] = \sigma_{av}[\psi]$. This proof avoids the additional assumption that $\psi(\alpha_1, \alpha_2)$ be single-valued in the MBZ. (Alternatively, if single-valuedness of $\psi(\alpha_1, \alpha_2)$ (hence of $\bar{\psi}(\alpha_1, \alpha_2)$) is assumed, $\gamma[\bar{\psi}; C]$ can be evaluated directly to give $2\pi$ times an integer.)

For completeness we briefly mention some further works on the topological approach to the IQHE, where the mathematical language differs from the one used in the original articles and where systems with substrate disorder and many-body interactions as in [4] are considered. References [10] investigate adiabatic charge transport, basically in a system with two “handles” (loops) enclosing two magnetic fluxes $\phi_1, \phi_2$, where $\phi_1 = -\frac{Vt}{\pi}$ is time-dependent. The Hall current is defined to be the current $I_2$ in loop no. 2 due to the electromotive force $V$. The Hall conductance is $I_2 / V$. The limit of weak $V$ (which is the limit of interest in linear response theory) is related to the adiabatic limit of the time-dependent Schroedinger equation. It is then shown (using projection operator methods) that, in this limit, the average of the Hall conductance over $0 \leq \phi_1 \leq \hbar c/e, 0 \leq \tau \leq \hbar c/(eV)$ leads to an expression of the type (2), which is equal to $e^2 / h \times$ an integer (a Chern number).

An analogous result has been obtained in [11]. Here, in the framework of non-commutative geometry, the Chern number could be related to the index of a Fredholm operator.

In two previous articles [7] (see also [9]) we have criticized the topological approach to the IQHE along two lines. First, from the fact that the averaging of the Kubo formula in [1, 3, 4] directly leads to an expression of type $\sigma_{av}[\bar{\psi}]$ (cf. (1), (2); (8), (12)), and observing that the parameter domain associated with the subspace of $\psi$ is not a torus (contrary to the claims made in [1] and [4]), we concluded that $\sigma_{av}[\bar{\psi}]$ is not a Chern number (times $e^2 / h$), hence not a topological invariant. This conclusion is erroneous, since $\sigma_{av}[\bar{\psi}]$ is equal to $\sigma_{av}[\psi]$, as we have seen above. Secondly, we also raised more physical arguments against the topological approach. These arguments remain.

The principle aim of the present article has been to explain the missing steps in the original papers [1, 4] on magnetic Bloch bands, by using the same language as in these original papers (we could not find this in the literature). This is important in view of the abundant theoretical literature based on these papers and also because of the spectacular physical predictions which they imply. For instance, according to the topological theory of $\sigma_{xy}$ presented in [1, 4], the individual Chern numbers associated with the subbands (magnetic Bloch bands) of a Landau band would imply the possibility of positive and negative(!) integer quantization of $\sigma_{xy}$ at fractional fillings of a Landau band. According to a recent investigation [12] these Chern numbers could be measured under certain conditions. However, such experiments have not yet been performed.
**A.1**

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**APPENDIX A**

From the definition of the unitary transformation $U(5)$ one obtains

$$U(k_1 L_1, k_2 L_2) \tilde{H}(\alpha_1, \alpha_2) U^{-1}(k_1 L_1, k_2 L_2) =$$

$$= (1/(2m))[-i\hbar \nabla + (\hbar/L)(k_1 L_1 + \alpha_1, k_2 L_2 + \alpha_2) - qA]^2 + V(x, y)$$

(which we denote by $\tilde{H}(k_1 L_1, k_2 L_2)(\alpha_2, \alpha_2)$),

and

$$U(k_1' L_1, k_2' L_2) \tilde{H}(\alpha_1 + 2\pi p_1, \alpha_2 + 2\pi p_2) U^{-1}(k_1' L_1, k_2' L_2) =$$

$$= (1/(2m))[-i\hbar \nabla + (\hbar/L)(k_1' L_1 + 2\pi p_1 + \alpha_1, k_2' L_2 + 2\pi p_2 + \alpha_2) - qA]^2 + V(x, y)$$

$$= \tilde{H}(k_1' L_1 + 2\pi p_1, k_2' L_2 + 2\pi p_2)(\alpha_2, \alpha_2).$$

If

$$k_1' L_1 = k_1 L_1 - 2\pi p_1, \quad k_2' L_2 = k_2 L_2 - 2\pi p_2, \quad p_1, p_2 \text{ integer},$$

the two Hamiltonians $\tilde{H}$ in (A.1) and (A.2) are equal (and further we have

$$\tilde{H}(\alpha_1 + 2\pi p_1, \alpha_2 + 2\pi p_2) =$$

$$= U(k_1 L_1 - k_1' L_1, k_2 L_2 - k_2' L_2) \tilde{H}(\alpha_1, \alpha_2) U^{-1}(k_1 L_1 - k_1' L_1, k_2 L_2 - k_2' L_2),$$

which shows that the spectrum of $\tilde{H}(\alpha_1, \alpha_2)$ is periodic in $\alpha_j, j = 1, 2$, with period $2\pi$.

We consider now the equation

$$\tilde{H}(\alpha_1, \alpha_2) \tilde{\psi}(k_1, k_2) = E(k_1, k_2)(\alpha_1, \alpha_2) \tilde{\psi}(k_1, k_2)$$

Since the spectrum of $\tilde{H}(\alpha_1, \alpha_2)$ is periodic in $\alpha_j, j = 1, 2$, with period $2\pi$ and continuous and discrete (except on isolated points of degeneracy), there exist wave vectors $k_1^*, k_2^*$, such that

$$\tilde{H}(\alpha_1 + 2\pi p_1, \alpha_2 + 2\pi p_2) \tilde{\psi}(k_1^*, k_2^*) = E(k_1^*, k_2^*)(\alpha_1 + 2\pi p_1, \alpha_2 + 2\pi p_2),$$

with

$$E(k_1^*, k_2^*)(\alpha_1 + 2\pi p_1, \alpha_2 + 2\pi p_2) = E(k_1, k_2)(\alpha_1, \alpha_2) = E.$$

We now apply the unitary transformation $U[k_1 L_1, k_2 L_2]$ to eq. (A.4) from the left and
U(k₁, L₁, k₂, L₂) to (A.5), respectively. This leads to the following two equations:

(A.6) \[ \hat{H}[k₁, L₁, k₂, L₂](α₁, α₂) U(k₁, L₁, k₂, L₂) \tilde{\psi}[k₁, k₂](α₁, α₂) = \]

\[ = E U(k₁, L₁, k₂, L₂) \tilde{\psi}[k₁, k₂](α₁, α₂) \]

and

(A.7) \[ \hat{H}[k₁, L₁ + 2πp₁, k₂, L₂ + 2πp₂](α₂, α₂). \]

\[ \cdot U(k₁, L₁, k₂, L₂) \tilde{\psi}[k₁, k₂](α₁ + 2πp₁, α₂ + 2πp₂) = \]

\[ = E U(k₁, L₁, k₂, L₂) \tilde{\psi}[k₁, k₂](α₁ + 2πp₁, α₂ + 2πp₂). \]

The two eigenfunctions in (A.6) and (A.7) belong to the same eigenvalue E. We now identify k₁, k₂ with k₁*, k₂* of (A.3). Then the two Hamiltonians \( \hat{H} \) in (A.6) and (A.7) are equal, see above. Therefore, and since the eigenvalues are discrete (except at isolated points of \((α₁, α₂)\)), the two eigenfunctions in (A.6) and (A.7) must now be equal (apart from a global phase factor), except at isolated points, which have to be crossed continuously. This means

(A.8) \[ U(k₁, L₁ + 2πp₁, k₂, L₂ + 2πp₂) \tilde{\psi}[k₁ + 2πp₁/L₁, k₂ + 2πp₂/L₂](α₁, α₂) \]

\[ \times \exp \{ i f(α₁, α₂; p₁, p₂) \} = U(k₁, L₁, k₂, L₂) \tilde{\psi}[k₁, k₂](α₁ + 2πp₁, α₂ + 2πp₂). \]

Here k_j has been expressed in terms of k_j' and p_j using (A.3). Relation (A.8) is equivalent to (10), since

\[ U(k₁, L₁ + 2πp₁, k₂, L₂ + 2πp₂) = U(k₁, L₁, k₂, L₂) U[2πp₁, 2πp₂]. \]

REFERENCES