On a set of equivalent theories to special relativity

G. Russo
Dipartimento di Fisica e Astronomia, Università di Catania
INFN - Sezione di Catania
Viale Andrea Doria, 6 - 95125 Catania, Italy

(ricevuto il 4 Novembre 2005; revisionato 24 Gennaio 2006; approvato il 25 Gennaio 2006)

Summary. — Following the guidelines drawn from Selleri, we propose an extended approach which does not use the assumption of slowing down of clocks in motion with respect to those in the preferred frame but solely the independence of the two-way speed of light over a closed path in any inertial frame. It results a set of equivalent theories to special theory of relativity depending on two free parameters, namely a synchronization parameter and a scaling factor. Then, in this context, we discuss the general problem on superluminal velocities and causal loop paradoxes connection for different values of the synchronization parameter.

PACS 03.30.+p – Special relativity.

1. – Introduction

Advances in technology have made it possible to carry out experimental tests of a fundamental postulate of the special theory of relativity (STR), the constancy of the in-vacuum velocity of light in all inertial reference frames. The idea goes back to the test theories of Robertson [1] and Mansouri and Sexl [2]. These are kinematical approaches to violations of special relativity based on the assumed existence of a preferred inertial reference system in which the speed of light is isotropic. In particular, by replacing Einstein’s postulates by facts drawn from experience, the so-called equivalent theories (ET) to the STR, based on a set of transformations depending on a synchronization parameter which is in large part conventional and often indicated as $e_1$, have been suggested [3]. Experimental tests [4] have shown that, when properly expressed in terms of measurable quantities, the results of such experiments are independent of the method chosen for the global synchronization of clocks. In the framework of the STR, recent theoretical [5] and experimental [6-9] evidences of superluminal motions necessarily lead to “unacceptable” causal loop paradoxes. The problem arises because while all observers agree about the

(*) The author of this paper has agreed to not receive the proofs for correction.
time ordering of events linked by a subluminal signal, for a superluminal(1) signal different observers disagree on whether the signal is received after or before it is emitted. In other words, viewed in a certain class of inertial frames, a superluminal signal travels backward in time. Thus, Tolman’s paradox [10], namely the communication with the past, would become possible in principle. In a previous work [11], we proposed a method to describe as, in the context of special theory of relativity, the existence of superluminal signals can lead to causal violations implying that cause and effect can be changed.

In this paper we present a similar method within weak relativity theories, otherwise known as equivalent theories to special relativity. In particular, after introducing a set of ET, based on the isotropy and invariance of the two-way light velocity for any inertial frame, we will show that for the subset of transformations characterized by $\varepsilon \geq 0$, the causal paradoxes are always forbidden. Within such a class of transformations, the one corresponding to the particular value $\varepsilon = 0$ provides the so-called “inertial transformations”, based, as is well known, upon the absolute simultaneity concept [12].

2. – The general transformation

It was shown by Selleri [3] that, by assuming the existence of a preferred frame $\Sigma_o$ in which Maxwell’s equations hold, implying that i) the vacuum velocity of light has the same value $c$ in all directions, and by adding the further hypothesis that ii) the space is homogeneous and isotropic and the time is homogeneous, one can always choose two systems of Cartesian orthogonal coordinates so that the transformation law from $\Sigma_o$ to an arbitrary inertial frame $\Sigma$ assumes the simple form

$$
\begin{align*}
x &= f_1(x_o - vt_o), \\
y &= g_2 y_o, \\
z &= g_2 z_o, \\
t &= e_1 x_o + e_4 t_o,
\end{align*}
$$

(1)

where the $f_1$, $g_2$, $e_1$, $e_4$ parameters are, generally, expected to be continuous functions of the velocity $v$ of the $\Sigma$ frame as measured in $\Sigma_o$. Moreover, as was shown in [3], the existence of the inverse transformation requires that $f_1$, $g_2$ and $R = e_1 v + e_4$ should be different from zero and in particular positive being $f_1(0) = g_2(0) = R(0) = 1$. As we will show in the following, the $e_1$ and $e_4$ parameters are depending on each other and such a dependence is related to the chosen synchronization method [2,13]. The $e_1$ to $e_4$ dependence should also assure the obvious independence of the synchronization method of the measured time, by means of one clock only, along each closed path in the $\Sigma$ frame. As an example in the standard Einstein synchronization we have $e_1 = -e_4 \beta / c$ (with the usual meaning for $\beta = v/c$) while in the case of synchronization by slow clock transport one obtains

$$
\frac{de_4}{dv} + v \frac{de_1}{dv} = 0.
$$

(1) Hereafter, by “superluminal” we always mean “with a speed higher than the speed of light in vacuum with respect to the considered inertial frame.
Nevertheless, being \( R(0) = 1 \), it should be always \( e_4(0) = 1 \) for any value of the \( e_1 \) parameter. When \( v \) goes to zero \( \Sigma_o \) and \( \Sigma \) become the same inertial frame but eqs. (1), with \( e_1 \) free parameter, do not reduce to the identity transformation but to a synchrony transformation with Anderson’s [14] parameter \( k = -\epsilon \). This shows that (1), in general, cannot form the Lorentz group, except for the special case \( \epsilon = 0 \) for \( v = 0 \). It is important to stress that the Lorentz transformations are a necessary consequence of Einstein’s postulates. In other words, the STR is “unstable”, in the sense that any shift, however small, of any one of the four coefficients \( f_1, g_2, e_4 \) and \( e_1 \), away from their relativistic values, necessarily implies the existence of a “preferred” frame. From the first and fourth of eqs. (1) we obtain

\[
(2) \quad t = R t_o + \frac{e_1}{f_1} x.
\]

Let now \( u_o(x_0) \) and \( u(x) \) be the velocities of a generic material particle (or of a signal) in the \( \Sigma_o \) and \( \Sigma \) frame, respectively, \( \Gamma \) a path which is supposed to be generally-regular and closed with respect to the \( \Sigma \) frame (the corresponding path \( \Gamma_o \) is obviously not closed for to the \( \Sigma_o \) frame and let \( A_o \) and \( B_o \) be its ordered extreme points). By indicating with \( s \) or \( s_0 \) the curvilinear abscissa in the \( \Sigma \) or \( \Sigma_o \) frame, from eq. (2) we obtain

\[
(3) \quad \oint_{\Gamma} \frac{ds}{u(x)} = R \oint_{\Gamma_o(A_o,B_o)} \frac{ds_0}{u_0(x_0)} + \frac{e_1 c}{f_1} \oint_{\Gamma} \frac{dx}{c}.
\]

Being \( \oint_{\Gamma} dx/c = 0 \), this result indicates both that the dimensionless quantity \( \epsilon = e_1 c / f_1 \) can be considered the suitable synchronization parameter and that \( R \) cannot depend on the chosen \( \epsilon \). \( R^{-1} \) should represent the “time-dilation” factor of clocks at rest in \( \Sigma \) with respect to those at rest in \( \Sigma_o \). Then, in the absence of further assumptions, the parameters which could be considered independent are \( f_1, g_2, R \) and \( \epsilon \). By choosing a suitable system of polar coordinates, the one-way velocity of light \( c_{\Sigma}(\theta) \) relative to \( \Sigma \), in a direction forming an angle \( \theta \) (as measured in \( \Sigma \)) with respect to the \( x \)-axis can be obtained from (1) as

\[
(4) \quad \frac{1}{c_{\Sigma}(\theta)} = \frac{1}{c} [\sigma \cos \theta + \mu (\cos^2 \theta + \lambda \sin^2 \theta)^{1/2}],
\]

where

\[
(5) \quad \mu = \gamma^2 \frac{R}{f_1} , \quad \sigma = \beta \mu + \epsilon , \quad \lambda = \left( \frac{f_1}{g_2 \gamma} \right)^2 ,
\]

and \( \gamma = (1 - \beta^2)^{-1/2} \). From eq. (4) we get the two-way velocity \( c_{fb} \) of light as

\[
(6) \quad \frac{c_{fb}}{c} = \mu^{-1} [1 - (1 - \lambda) \sin^2 \theta]^{-1/2}.
\]

Because of the space isotropy in \( \Sigma_o \) the transformation \( \vec{r} \to \vec{r}' = -\vec{r} \) has to hold under both \( \vec{r}_o \to \vec{r}_o' = -\vec{r}_o \) and \( \vec{v} \to \vec{v}' = -\vec{v} \) parity transformations. This requires \( f_1, g_2, e_4 \) to be even functions, and \( e_1 \) to be an odd function of \( v \). Thus, from eqs. (5) we conclude that \( \sigma \) is odd while \( \mu \) and \( \lambda \) are even functions of \( v \). For small values of \( \beta \), from
McLaurin expansion, being their odd order derivates odd functions of \( v \), we shall assume that in (6) \( \mu \) and \( \lambda \) are functions of \( \beta^2 \). Thus, by setting \( \alpha \equiv \beta^2 \) we obtain

\[
\frac{\Delta c}{c} = \beta^2 [A + B \cos^2 \theta]
\]

with

\[
B = \left( \frac{\partial \lambda}{\partial \alpha} \right)_{\alpha=0}, \\
A = -B + \left( \frac{\partial \mu^{-1}}{\partial \alpha} \right)_{\alpha=0}.
\]

This result is in agreement with a commonly adopted way to model a violation of STR in the framework of the test theory of Robertson [1], further extended by Mansouri and Sexl [2] (RMS theory). According to [1] and [2], the STR relies on a threefold experimental basis. This means that three experiments are necessary to provide a complete test of STR: i) a test of the isotropy of the vacuum speed of light (Michelson-Morley [15] type), ii) a test of the independence of the velocity of light from that of the source (Kennedy-Thorndike [16] type) and iii) a test of the time dilation factor (Ives-Stilwell [17] type). Concerning modern versions of the above-mentioned historical experiments, we direct the reader to the improved tests of [18, 19] and [20], respectively. Assuming a preferred frame of reference (e.g., the Cosmic Microwave Background) the RMS theory provides the prediction of a hypothetical anisotropy and velocity dependence of the vacuum speed of light which is the same of eq. (7). Indeed, the general transformation (1) and (2) provides the Mansouri and Sexl [2] one by setting \( a = R, b = f_1, d = g_2 \) and Mansouri’s synchronization parameter \( \epsilon_M = \epsilon/c \). In order to compare our parametrization of synchronization also with the early Reichenbach’s one (but operationally equivalent) [21] let us consider the special cases of forward and backward light propagation with respect to the \( \Sigma \) frame. From eq. (4), we obtain

\[
\frac{1}{c(0)} - \frac{1}{c(\pi)} = \frac{2}{c} \sigma,
\]

which in the \( \beta \to 0 \) limit, leads to the equation \( \epsilon = 2\epsilon_R - 1 \) where we have indicated with \( \epsilon_R \) the usual conventional parameter (restricted to the open interval between 0 and 1) as introduced by Reichenbach [21] and corresponding to alternative stipulations concerning the one-way speed of light.

An argument against the conventionality thesis has been given by Malament [22], who argues that standard synchrony is the only simultaneity relation that can be defined from the symmetric causal connectibility. Sarkar and Stachel [23] argue there is no physical warrant for the requirement that a simultaneity relation be invariant under temporal reflections. Thus, dropping that requirement, they show that Malament’s other criteria for a simultaneity relation can be still satisfied. Giulini [24], considering Sarkar and Stachel’s arguments to be mathematically incomplete, examines the general problem of uniqueness of certain simultaneity structures in flat space-time and concludes that the Einstein standard synchrony is the unique relative simultaneity when the structure is taken to be a foliation of space-time by straight lines.
Presently, the debate about conventionality of simultaneity seems far from to be settled, although some proponents of both sides of the argument might disagree with that statement. For a detailed analysis about the conventionality of synchronization in relation to the interpretation of recent experimental test of special relativity we direct the reader to the exhaustive review papers of Anderson [14] and Lämmerzahl [25].

Finally, we remark that in the STR one has $g_2 = 1$, $f_1 = \gamma$, $R = \gamma^{-1}$ (and consequently gets $A = B = 0$) and $\epsilon = -\beta$.

3. – Theories equivalent to special relativity

It is well known that one of the most fundamental ideas underlying the conceptual edifice of special relativity is the independence of the one-way velocity of light from its direction of propagation. This represents a useful conventional assumption but, as some recent authors [2,13] have observed, it is not logically necessary. For instance, Mansouri and Sexl [2] observed that when clocks are synchronized according to Einstein’s procedure, the equality of the velocity of light in two opposite directions is a consequence of synchronization and cannot really be checked with experiments. On the other hand, the one-way velocity of light has never been measured accurately [26]. Measurements which depend on a direct, one-time comparison of separated clocks, i.e. by a time-of-flight technique between two points, will depend on the synchronization procedure of the two clocks. However, a test of the isotropy of the speed of light between the same two clocks as the orientation of the propagation path varies relative to the reference frame should not depend on how they were initially synchronized. Very accurate measurements of the two-way velocity of light are obviously possible using only one clock as the synchronization problem does not arise. Thus, the invariance of the one-way velocity of light being only a convention [21], theories based on weaker postulates have been developed [3]. In particular, following the idea of replacing Einstein’s postulates by facts drawn from experience, the so-called “equivalent transformation” was obtained by Selleri [3] by adding two well-established empirical results, namely i) the independence of $c_{fb}$ from $\theta$ and $\beta$ (up to now the experimental tests of (7) have provided very small values for $A$ and $B$); ii) the slowing down of clocks at rest in $\Sigma$ with respect to those in $\Sigma_o$.

However, concerning the ii) postulate we should observe that it is based on a misinterpretation of experimental evidences. The above space and time transformations regard a generic $\Sigma$ frame and the $\Sigma_o$ preferred frame. Then, the time dilation represented by the $R^{-1}$ parameter is actually independent of the synchronization parameter $\epsilon$ due to the assumption for the isotropic one-way light velocity in the $\Sigma_o$ frame. But, the experiments provide time interval measurements between different inertial frames and not between a generic inertial frame and the preferred one. Moreover, as outlined also by Lämmerzahl [25], the time dilation factor should necessarily depend on the synchronization procedure and therefore cannot be used as experimental irrefutable evidence because, strictly speaking, it is not directly measurable.

For this reason, we abandon Selleri’s ii) postulate and will derive the transformation based on the i) assumption only. It is straightforward to deduce that the hypothesis i) implies $\lambda = \mu = 1$. In fact, we have

$$\frac{\partial}{\partial \theta} \left[ \frac{1}{c_{fb}(\theta)} \right] = \frac{\mu}{2c} \left[ 1 - (1 - \lambda)\sin^2 \theta \right]^{-1/2} \sin(2\theta)(\lambda - 1) = 0, \quad \forall \theta,$$

which gives $\lambda = 1$. By substitution in (6) we get from i) also $\mu = 1$. Let us now
demonstrate that the i) assumption is equivalent to assume that, for an arbitrary inertial frame \( \Sigma \), the time interval of the light in a closed path (deduced from (4)) is just equal to that taken, within the STR, i.e. when its velocity is assumed equal to \( c \) for any direction. This means to impose the following condition:

\[
\oint_{\Gamma} \frac{ds}{c} = \oint_{\Gamma} \frac{ds}{c}
\]  

for any generally-regular and closed curve \( \Gamma \). In fact, let us consider a generic closed curve \( \Gamma \) and let

\[
x = x(t), \quad y = y(t), \quad z = z(t)
\]

be a possible parametric representation of it. As we always have

\[
\oint_{\Gamma} \sigma \cos \theta ds = \sigma \oint_{a}^{b} \frac{x'(t)}{H(t)} dt = \sigma [x(t)]_{a}^{b} = 0
\]

with

\[
H(t) = [x'^2(t) + y'^2(t) + z'^2(t)]^{1/2}, \quad \forall t \in [a,b],
\]

eq. (11) implies that, for any \( \Gamma \) curve, the following equation should be satisfied:

\[
\int_{a}^{b} \{ \mu [x'^2(t) + \lambda y'^2(t) + \lambda z'^2(t)]^{1/2} - H(t) \} dt = 0.
\]

Following [3] we recall that \( \mu \) and \( \lambda \) cannot assume the zero value. Moreover, because of the chosen orientation of the spatial axes, they should be both positive. We observe that \( \lambda = 1 \) and \( \mu = 1 \) is certainly a solution of eq. (15) for any generally-regular and closed curve \( \Gamma \). In order to prove that no other solutions satisfying the same requirement exist, let us define the following function:

\[
f_{\Gamma}(\lambda, \mu) = \int_{a}^{b} \{ \mu [x'^2(t) + \lambda y'^2(t) + \lambda z'^2(t)]^{1/2} - [x'^2(t) + y'^2(t) + z'^2(t)]^{1/2} \} dt
\]

and examine the equation

\[
f_{\Gamma}(\lambda, \mu) = 0, \quad \text{for} \quad \lambda > 0, \quad \mu > 0, \quad \text{and} \quad \forall \Gamma.
\]

This latter represents a set of curves in the \( (\lambda, \mu) \)-plane corresponding to generally regular and closed curves in the \( x, y, z \) space. We conjecture that such curves have in common the \( (1,1) \) point only.
For that, it is enough to show that we can find in the $x, y, z$ space two curves $\Gamma_1$ and $\Gamma_2$ which correspond, in the $(\lambda, \mu)$-plane, to two curves $f_{\Gamma_1}(\lambda, \mu) = 0$ and $f_{\Gamma_2}(\lambda, \mu) = 0$ having in common the (1,1) point only. In fact, by choosing $\Gamma_1$ to be any curve in a plane parallel to $x = 0$, we get the equation

\[ \mu \sqrt{\lambda} = 1, \tag{17} \]

while, by choosing, in the $z = 0$ plane, a triangular curve delimited by the straight lines having $y = x$, $x = 1$ and $y = 0$ equations, we obtain

\[ \mu (1 + \sqrt{\lambda} + \sqrt{1 + \lambda}) = 2 + \sqrt{2} \tag{18} \]

and these have in common, in the $\lambda > 0$, $\mu > 0$ quadrant, the (1,1) point only. By setting $\lambda = \mu = 1$, we find from (5)

\[ f_1 = \gamma g_2, \quad R = \frac{g_2}{\gamma}, \quad \sigma = \beta + \epsilon. \tag{19} \]

Equation (13) still indicates that the time it takes the light for a fixed round-trip does not depend on the $\sigma$ value which, therefore, has to be considered a function of the synchronization parameter. Finally, the set of ET transformations writes

\[ x = g_2 \gamma (x_o - \beta ct_o), \]
\[ y = g_2 y_o, \]
\[ z = g_2 z_o, \]
\[ ct = \frac{g_2}{\gamma} ct_o + \epsilon x = g_2 \gamma [(1 - \beta^2 - \beta \epsilon) ct_o + \epsilon x_o], \tag{20} \]

while the inverse ones are

\[ x_o = \frac{\gamma}{g_2} [(1 - \beta^2 - \beta \epsilon) x + \beta ct], \]
\[ y_o = \frac{1}{g_2} y, \]
\[ z_o = \frac{1}{g_2} z, \]
\[ ct_o = \frac{\gamma}{g_2} (ct - \epsilon x). \tag{21} \]

From eqs. (4) and (19) we get the ET anisotropy prediction for the in-vacuum one-way light velocity in the $\Sigma$ reference frame

\[ \beta_\Sigma(\theta) = \frac{c_\Sigma(\theta)}{c} = \frac{1}{1 + (\beta + \epsilon) \cos \theta} \tag{22} \]

which is independent of the $g_2$ parameter. It is possible to show that the transformation laws (20) can explain the available experimental evidences in spite of the implied noninvariance of the in vacuum one-way velocity of light equation (22). In particular, it results
that the main experimental evidences, usually called upon to support the STR, are actually insensitive to the choice of the $g_2$ and $\epsilon$ parameters. It is worth noting that eq. (11) provides the reason why such an insensitivity is expected in all experiments which use closed paths for the light. Moreover, eq. (22) explains also why the difference of time measured along two open path between two fixed points, is completely independent of the $\sigma$ value (and consequently of the synchronization parameter $\epsilon$) resulting the same of that provided in the STR [3, 26]. Recently [27], it was shown that the reflection law of the light from a uniformly moving plane mirror is a direct consequence of the constant one-way light speed postulate. For this purpose, we would like to remark that such a law can be also deduced from (21), resulting quite independently from a particular choice for the $\epsilon$ and $g_2$ parameters.

4. – Causal loop paradoxes

At first, we remark that each equation we are deducing in this section will give the previous one provided in [11] when we will substitute in it $\epsilon = -\beta$ and $g_2 = 1$. As we are interested in the study of violation of the “spatially coincident causality”, in the following we will confine our consideration only to the case $|\sigma| < 1$ which corresponds to assume $\epsilon$ restricted to the open interval between $-1-\beta$ and $1-\beta$. Let us suppose now that a process exists in which an event $F$ causes an event $G$ at a “superlight” speed $u_o > c$ relative to the preferred frame $\Sigma_0$. Let us choose coordinates in $\Sigma_0$ so that both events occur on the $x_o$-axis, and let their spatial and time separations be $\Delta x_o > 0$ and $\Delta t_o > 0$. Then, in another inertial frame $\Sigma$ we have from the fourth of eqs. (20),

$$\Delta t = g_2 \gamma \Delta t_o \left[1 - \beta^2 + \epsilon \left(\frac{u_o}{c} - \beta\right)\right].$$

(23)

For $\epsilon \geq 0$, $\Delta t$ and $\Delta t_o$ have the same sign.

For $\epsilon < 0$ and

$$c/u_o < -\frac{\epsilon}{1 - \beta^2 + \epsilon\beta}$$

(24)

we have $\Delta t < 0$. This implies the existence of a class of inertial frames $\Sigma$ in which $G$ precedes $F$, i.e. in which cause and effect are reversed or in which information flows from receiver to transmitter. Thus, we could have foreknowledge of future events and if we decide to deliberately foil them, by manipulating the past, we would incur serious contradictions.

Let us consider, at the time $t(F_1)$, a superluminal signal relative to $\Sigma$ (having a velocity $\tilde{\beta}$ in $c$ units), emitted from its origin towards the origin of $\Sigma_0$. $G_1$ is the event associated to its arrival in $\Sigma_0$. The observer in $\Sigma_0$, after waiting for a time $\Delta t_o > 0$, decides to send, toward the origin of $\Sigma$, a superluminal signal, relative to $\Sigma_0$ (event $G_2$). $F_2$ is the event associated with the arrival of the signal in the origin of $\Sigma$. The only constraint is that each signal should travel in the future with respect to the observer which has emitted it. Well, now we shall study under which conditions for the superluminal velocities non-negative solutions exist for $\Delta t_o$, for which the event $F_2$ happens before the $F_1$ one with respect both to the $\Sigma_0$ and $\Sigma$ reference frames. Indeed, we have for the signal starting from $F_1$

$$x(G_1) = -\tilde{\beta}[ct(G_1) - ct(F_1)]$$

(25)
and also from eqs. (20) or from the first of eqs. (21),

\begin{equation}
    x(G_1) = \frac{-\beta}{1 - \beta^2 - \epsilon \beta} ct(G_1),
\end{equation}

\( G_1 \) being an event which happens on the \( x_o = 0 \) axis. Thus, being

\begin{equation}
    \tilde{\beta} > \beta_\Sigma(\pi) > \frac{\beta}{1 - \beta^2 - \epsilon \beta},
\end{equation}

where the last term represents the velocity (in \( c \) units) of the \( \Sigma_o \) frame with respect to the \( \Sigma \) one, we get

\begin{equation}
    ct(G_1) = \frac{\tilde{\beta}}{\beta - \frac{\tilde{\beta}}{1 - \beta^2 - \epsilon \beta}} ct(F_1)
\end{equation}

and, using the fourth equation of the ET transformation (20), we find the result

\begin{equation}
    ct_o(G_1) = \frac{1}{\gamma g_2} \frac{\tilde{\beta}}{\beta(1 - \beta^2 - \epsilon \beta) - \beta} ct(F_1).
\end{equation}

After a time interval \( \Delta t_o \geq 0 \), the observer in \( \Sigma_o \) sends, from its origin, a superluminal signal (having a velocity \( \beta^* \) in \( c \) units) and let \( G_2 \) be the event associated with its departure. Thus, if \( F_2 \) denotes the event associated with the arrival of such a signal in the origin of \( \Sigma \), we have

\begin{equation}
    t_o(G_2) = t_o(G_1) + \Delta t_o,
\end{equation}

\begin{equation}
    x_o(F_2) = \beta^* [ct_o(F_2) - ct_o(G_2)],
\end{equation}

\begin{equation}
    x_o(F_2) = \beta ct_o(F_2).
\end{equation}

Therefore, being

\begin{equation}
    \beta^* > 1 > \beta,
\end{equation}

we get

\begin{equation}
    ct_o(F_2) = \frac{\beta^*}{\beta^* - \beta} ct_o(G_2)
\end{equation}

and using the fourth equation of the inverse ET transformation (21) (being \( x(F_2) = 0 \)) we find the result

\begin{equation}
    ct(F_2) = \frac{g_2 \beta^*}{\gamma(\beta^* - \beta)} ct_o(G_2).
\end{equation}
By using eqs. (29) and (30), eq. (35) provides

\[ ct(F_2) = \frac{\beta^*}{\gamma(\beta^* - \beta)} \left\{ g_2 e^{\Delta t_o} + \frac{\tilde{\beta}}{\gamma[\beta(1 - \beta^2 - \epsilon\beta) - \beta]} ct(F_1) \right\}. \]  

(36)

Now, for non-negative values of $\Delta t_o$, the $ct(F_2) < ct(F_1)$ inequality necessarily requires that the condition

\[ \frac{1}{\gamma^2} \frac{\tilde{\beta} \beta^*}{\tilde{\beta}(1 - \beta^2 - \epsilon\beta) - \beta(\beta^* - \beta)} < 1 \]  

(37)

should be, at least, satisfied. This can be written as

\[ \beta^* + (1 - \beta^2 - \epsilon\beta) \tilde{\beta} + \epsilon \tilde{\beta} \beta^* - \beta < 0. \]  

(38)

At first let us examine the $\epsilon = 0$ case. In this case the inequality (38) writes

\[ \beta^* + (1 - \beta^2) \tilde{\beta} < \beta, \]  

(39)

which is never satisfied resulting in contradiction with inequalities (27) and (33) which provide

\[ \beta^* + (1 - \beta^2) \tilde{\beta} > 2\beta. \]  

(40)

For $\epsilon \neq 0$, the condition (38) has an interesting geometrical representation in the $(\tilde{\beta}, \beta^*)$-plane. In fact, it represents a region delimited by a hyperbola which, by means of the translation

\[ \tilde{\beta} = \Lambda - \frac{1}{\epsilon}, \]  

(41)

\[ \beta^* = \Lambda^* - \frac{1}{\epsilon}(1 - \beta^2 - \epsilon\beta), \]  

(42)

becomes

\[ \tilde{\Lambda} \Lambda^* > \frac{1}{(\epsilon \gamma)^2}, \quad \text{for} \quad \epsilon < 0 \]  

(43)

and

\[ \tilde{\Lambda} \Lambda^* < \frac{1}{(\epsilon \gamma)^2}, \quad \text{for} \quad \epsilon > 0. \]  

(44)

By using inequalities (27) and (33), translations (41) and (42) provide

\[ \Lambda > \Lambda_o \equiv \frac{1 - \beta}{\epsilon(1 - \beta - \epsilon)}, \]  

(45)
Fig. 1. – Graphic representation of inequalities (43) and (44) in the $\tilde{\Lambda} - \Lambda^*$ plane. The shadowed region indicates the set of values of $\tilde{\Lambda}$ and $\Lambda^*$ for which causal paradoxes become possible.

\begin{equation}
\Lambda^* > \Lambda^*_0 \equiv 1 - \beta + \frac{1}{\gamma^2}.
\end{equation}

Moreover, as the quantity $\tilde{\Lambda}_o\Lambda^* - \frac{1}{(\epsilon\gamma)^2}$ has the same sign of $\epsilon$, the point $P_o$ whose coordinates are $(\tilde{\Lambda}_o, \Lambda^*_o)$ lies in the region opposite to that delimited by the inequalities (43) and (44). As a consequence of that, we have no solution in the $\epsilon > 0$ case (i.e., causal paradoxes are forbidden) while for $\epsilon < 0$ we find that causal paradoxes are permitted. In fig. 1 we show the region characterized from $\tilde{\Lambda} > 0$ and $\Lambda^* > 0$ inequalities, in which eq. (43) is satisfied, namely the region (shadowed) where causality can be violated.

Finally, under condition (43) we find an interval of non-negative values for $\Delta t_o$ which is limited by a maximum value given by

\begin{equation}
\Delta t_{o,\text{max}} = \frac{-c\gamma\beta t(F_1)}{\beta^*[\beta(1 - \beta^2 - \epsilon\beta) - \beta]} \left\{ (\beta + \frac{1}{\epsilon})[\beta^* + \frac{1}{\epsilon}(1 - \beta^2 - \epsilon\beta)] - \frac{1}{(\epsilon\gamma)^2} \right\}.
\end{equation}

\begin{equation}
= \frac{-c\gamma^2\beta}{\beta^*\beta} t_o(G_1) \left[ \tilde{\Lambda}\Lambda^* - \frac{1}{(\epsilon\gamma)^2} \right].
\end{equation}

5. – Conclusions

In conclusion, in this paper we have shown the general transformation of the space-time coordinates between the so-called preferred frame and an arbitrary inertial frame. We have discussed the connection of our approach with the test theories of Robertson [1]...
G. RUSSO

and Mansouri and Sexl [2]. Moreover, by following Selleri’s [3] idea of replacing Einstein’s postulates by facts drawn from experience, we have proposed an extended approach which provides a set of theories which are equivalent to the special theory of relativity. As our approach does not use the hypothesis of the slowing down of clocks in motion with respect to those in the preferred frame, but solely the independence of the two-way speed of light over a closed path in any inertial frame it results to be more general than Selleri’s one. In fact, our set of equivalent theories is depending on two free parameters, namely a synchronization parameter \( \epsilon \), defined in order to assure the independence on time measurements requiring only one clock from its value and a scaling factor \( g \). In the context of such a more general framework we have shown that, for the subset of ET characterized from a synchronization parameter \( \epsilon < 0 \), we can incur grave causal paradoxes when the velocities of signals (in \( c \) units), declared “superluminal” with respect to each inertial observer, are greater than

\[
-\frac{1}{\epsilon} \left(1 - \beta^2 - \epsilon \beta\right) \quad \text{and} \quad -\frac{1}{\epsilon}
\]

for \( \Sigma_o \) and \( \Sigma \) frames, respectively, whereas, for the subset of transformations characterized by \( \epsilon \geq 0 \), the causal paradoxes are avoided.

REFERENCES