Multimomentum maps on null hypersurfaces

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Summary. — This paper studies the application of multimomentum maps to the constraint analysis of general relativity on null hypersurfaces. It is shown that, unlike the case of spacelike hypersurfaces, some constraints which are second class in the Hamiltonian formalism turn out to contribute to the multimomentum map. To recover the whole set of secondary constraints found in the Hamiltonian formalism, it is necessary to combine the multimomentum map with those particular Euler-Lagrange equations which are not of evolutionary type. The analysis is performed on the outgoing null cone only.

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1. - Introduction

In the Hamiltonian formulation of general relativity, the constraint analysis on null hypersurfaces plays an important role since such surfaces provide a natural framework for the study of gravitational radiation in asymptotically flat space-times [1-6]. Moreover, in a null canonical formalism, the physical degrees of freedom and the observables of the theory may be picked out more easily [4, 5].

On the other hand, relying on the multysymplectic formalism for classical field theories described, for example, in ref. [7], recent work in the literature [8-10] has studied the formulation of general relativity in terms of jet bundles. In this formalism, the local description involves local coordinates on Lorentzian space-time, tetrads, connection one-forms, multivelocities corresponding to the tetrads and multivelocities corresponding to the connection one-forms. The derivatives of the Lagrangian with respect to the latter class of multivelocities give rise to a set of multimomenta which naturally occur in the constraint equations. All the constraint equations of general relativity are then found to be linear in terms of this class of multimomenta. In ref. [9], the construction of ref. [8] has been extended to complex general relativity, where Lorentzian space-time is replaced by a four-complex-dimensional complex-Riemannian manifold. One then finds a holomorphic theory where the familiar constraint equations
are replaced by a set of equations linear in the holomorphic multimomenta, provided that such multimomenta vanish on a family of two-complex-dimensional surfaces [9, 10].

In the light of the properties and results briefly outlined, we have been led to consider the Lagrangian version of a constraint analysis on null hypersurfaces, when the multisymplectic formalism [7] is applied. For this purpose, sect. 2 describes null tetrads, while the analysis of multimomentum maps on null hypersurfaces is performed in sect. 3. Self-dual gravity is studied in sect. 4, and concluding remarks are presented in sect. 5.

2. Null tetrads

In this paper we are only interested in a local analysis of null hypersurfaces. Thus, many problems arising from the possible null-cone singularities are left aside. To give a geometric description of a null hypersurface, it is possible to introduce, as in ref. [4], a null tetrad with components

\[ e_0 = \frac{1}{N} \left( \frac{\partial}{\partial t} - N^i \frac{\partial}{\partial x^i} \right), \]

and

\[ e_k = - \frac{\alpha \bar{k}}{N} \frac{\partial}{\partial t} + \left( \nu^i + \alpha \bar{k} \frac{N^i}{N} \right) \frac{\partial}{\partial x^i}, \]

where \( N \) is the lapse function and \( N^i \) are components of the shift vector. The duals to (2.1) and (2.2) are

\[ \theta^0 = (N + \alpha, N^i) \, dt + \alpha_i \, dx^i \]

and

\[ \theta^k = \nu^i (N^i \, dt + dx^i), \]

where tetrad labels \( \bar{a}, \bar{b}, \bar{c} = 0, 1, 2, 3 \), while the indices \( \bar{k}, \bar{l} = 1, 2, 3 \). Analogous notation is used for the space-time indices \( a, b, \ldots \) and \( i, j \). Moreover, one has

\[ \nu^i = \delta^i_{\bar{k}}, \]

and

\[ \alpha \bar{k} = \nu^i \alpha_i. \]

Given the metric defined by

\[ \eta_{\bar{a}\bar{b}} = \eta^{\bar{a}\bar{b}} \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \]
the space-time metric can be expressed as \( g = \eta_{\theta^a \theta^b} \). It is then straightforward to see that, on the hypersurfaces defined locally by the equation \( t = \text{constant} \), one has

\[
g^{ab} t_a t_b = -\frac{2}{N^2} (\alpha_1 + \alpha_2 \alpha_3).
\]

This implies that such hypersurfaces are null if and only if

\[
\alpha_1 + \alpha_2 \alpha_3 = 0.
\]

By a particular choice of coordinates, it is always possible to set \( \alpha_1 = \alpha_2 = 0 \) \[4\].

In most of the following equations the tetrad vectors appear in the combination

\[
p_{\alpha_c}^{ab} = \frac{e_a}{2} (e^b_{\alpha_c} - e^b_{\xi} e^\xi_{\alpha_c}),
\]

where \( e = N \nu \) with \( \nu = \det(N^\nu) \). In a covariant Hamiltonian version of the theory, these quantities can be identified with the multimomenta introduced in refs. \[8, 10\].

3. Multimomentum maps on null hypersurfaces

The multimomentum map is a geometric tool which encodes the relevant information about the invariance properties of a classical field theory and its first-class constraints \[7-10\]. Indeed, the terminology used so far by the authors \[8-10\] differs from the one in ref. \[7\]. As far as we can see, what we call multimomentum map corresponds to the \textit{energy-momentum map} defined in ref. \[7\].

In particular, in general relativity, the evaluation of the multimomentum map on a spacelike hypersurface \( \Sigma \) can be expressed in terms of the following integral \[8\]:

\[
I_\Sigma \big[ \xi, \lambda \big] = \int_{\Sigma} \left[ \tilde{p}^{abcd} (\xi^b a \omega^d c - (D_a \lambda)^d b + \omega^d a \xi^b c) + \frac{1}{2} \tilde{p}^{abd c} \Omega_{ab}^{cd} \xi^c \right] d^3 x_c.
\]

With our notation, \( \xi \) is a vector field describing infinitesimal diffeomorphisms on the base space (i.e. space-time), \( \omega^a_{bc} \) are the connection one-forms and \( \Omega_{ab}^{cd} \) are the curvature two-forms. Moreover, the antisymmetric \( \lambda^{ab} \) is an element of the algebra \( o(3, 1) \), and \( D_a \) denotes covariant differentiation with respect to a Lorentz connection which annihilates the Minkowskian metric of the internal space \[8\]. The key point of our analysis is now the evaluation of the integral (3.1) on a null hypersurface \( J_{\nu} \), and then the interpretation of the resulting contributions in terms of a subset of the constraint equations. Indeed, by virtue of the formalism described in sect. 2, the multimomenta on a null hypersurface read

\[
\tilde{p}^{0i} = \frac{e}{2N} V^i,
\]

\[
\tilde{p}^{ij} = -\frac{e}{N} V^i V^j,
\]

\[
\tilde{p}^{0i} = \frac{e}{N} V^k \alpha_{ik} = 0,
\]

\[
\tilde{p}^{ij} = e V^k V^j + \frac{e}{N} V^k \alpha_{ij} N^j = e V^k V^j,
\]
where we have used the freedom to set to zero two of the $\alpha$ parameters, jointly with eq. (2.9). We now integrate by parts in eq. (3.1) and we impose the boundary conditions of ref. [8], according to which the multimomenta or the gauge parameters $\xi^a$ and $\lambda^{ab}$ should vanish at the boundary $\partial \Sigma$. Moreover, we restrict ourselves to the adapted coordinates for null hypersurfaces (cf. [8]), which implies that only the integration $d^3x_0$ survives in eq. (3.1). This means that the integration is only performed on the outgoing null cone [1, 2].

Thus, in the light of eqs. (3.2)-(3.5), on setting to zero the multimomentum map on a null hypersurface (this is what one does on spacelike hypersurfaces to obtain the constraints [7]) one finds the equations

\[ \int_{\Sigma'} \lambda^{i k} \left[ \partial_i \left( \frac{e}{N} V^l_k \right) + \frac{e}{N} \omega_{i k} V^l_{i j} \right] d^3 x_0 = 0 , \]

\[ \int_{\Sigma'} \lambda^{i k} \left[ \partial_i \left( \frac{e}{N} V^j_{i k} \alpha_{i j} \right) + \frac{e}{2 N} \left[ \omega_{i k} V^l_{i j} - \omega_{i l} V^j_{i k} \right] \right] d^3 x_0 = 0 , \]

\[ \int_{\Sigma'} \left[ e V^l_i V^j_{i k} \Omega_{i j}^{k l} - \frac{2 e}{N} \left[ \Omega^l_{i j} \Omega^{k l}_{i j} \right] \right] d^3 x_0 = 0 , \]

\[ \int_{\Sigma'} \left[ \frac{e}{N} V^j_{i k} \Omega_{i j}^{k l} \xi^l \xi^j \right] d^3 x_0 = 0 . \]

However, eqs. (3.6)-(3.9) are only a subset of the full set of constraints in the theory (see below).

Indeed, the Palatini action

\[ S_p = \frac{1}{2} \int d^4 x \, e e^a_b e^b_c \Omega_{ab} \Omega_{bc} , \]

leads to the Euler-Lagrange equations [8]

\[ G^\xi \equiv e^b_c \left[ \Omega_{\xi b} - \frac{1}{2} e^d_{\xi} e^e_{\xi} \Omega_{\xi d} \right] = 0 , \]

and

\[ D_{\xi} \bar{\rho}_{ab} = 0 . \]

On using eqs. (3.2)-(3.5), it is then possible to show by inspection that the complete set of equations corresponding to the secondary constraints of the Hamiltonian formalism are (3.6)-(3.9), and the nine equations

\[ D_{\xi} \bar{\rho}_{ab} = 0 , \quad i, j \neq 0 , \]
since these equations do not depend on time derivatives when the $\alpha_i$ are set to zero (see eq. (3.4)).

Thus, the multimomentum map does not provide all the constraints, but only a subset of them. To make further progress, it is necessary to compare the set of constraints obtained here with those found in the corresponding Hamiltonian approach [4, 6].

4. - Self-dual gravity

In refs. [4, 6] the Hamiltonian formulation of a complex self-dual action on a null hypersurface in Lorentzian space-time was studied. The $3+1$ decomposition was inserted into the Lagrangian, and the constraints were derived with the usual Dirac procedure. In this section the results of ref. [4] are briefly summarized and then compared with the corresponding constraints obtained by the multimomentum map. Since the constraints found in Lagrangian formalism correspond to the secondary constraints of the Hamiltonian formalism [11], the discussion is focused on these ones.

The complex self-dual part of the connection are the complex one-forms given by

$$\omega^a = \frac{1}{2} \left( \omega^a_{\dot{a}\dot{c}} - \frac{i}{2} \varepsilon^{\dot{a}\dot{c}\dot{d}} \omega^a_{\dot{d}} \right).$$

Explicitly, one has

$$\omega^{a0} = \frac{1}{2} \omega^{a0} + \omega^{a2}, \quad \omega^{a1} = \omega^{a2}, \quad \omega^{03} = \omega^{03} = \omega^{02} = \omega^{13} = 0.$$

The curvature of a self-dual connection is equal to the self-dual part of the curvature:

$$\Omega^{(+1)} = \Omega^{(-1)}(\omega).$$

Thus, the complex self-dual action to be considered is [8]

$$S_{SD} = \frac{1}{2} \int d^4x e_a e^{b\dot{a}} \Omega_{ab}.$$
The irreducible second-class constraints turn out to be $\mathcal{H}_0$, $\mathcal{H}_3$, $\chi'$, $\phi_1 V_2^\perp$ and $\phi_1 V_3^\perp$ [4]. Note that, following refs.[4, 6], we have set to zero all the $\alpha$ parameters in the course of deriving eqs. (4.6)-(4.12).

Let us now discuss the constraints from the Lagrangian point of view. The multimomentum map is formally the same as in the Palatini case, provided that

\begin{equation}
(4.1) \quad \mathcal{H}_2 \equiv - \frac{e}{N} \left[ \gamma^0 \omega^1_1 V_2^\perp \right] = 0,
\end{equation}

\begin{equation}
(4.10) \quad \mathcal{H}_3 \equiv - \partial_i \left( \frac{e}{N} V_3^\perp \right) + \frac{e}{N} \gamma^1 \omega^2_1 V_1^\perp + 2 \frac{e}{N} \gamma^0 \omega^3_1 V_3^\perp = 0,
\end{equation}

\begin{equation}
(4.11) \quad \chi' \equiv - 2 \partial_i \left( \frac{e^2}{N} V_2^\perp \right) - 2 \frac{e^2}{N} \gamma^0 \omega^3_1 V_2^\perp + \frac{e^2}{N} \gamma^1 \omega^0_1 V_1^\perp +
\end{equation}

\begin{equation}
+ 2 \frac{e}{N} \gamma^0 \omega^3_1 V_1^\perp + \frac{e}{N} \gamma^0 \omega^0_1 V_3^\perp = 0,
\end{equation}

\begin{equation}
(4.12) \quad \phi_1 \equiv - \frac{e}{N} \left[ \gamma^0 \Omega_{ij}^1 \omega^2_1 V_2^\perp + \gamma^0 \Omega_{ij}^2 V_2^\perp \right] = 0.
\end{equation}

The constraint equations obtained from setting to zero this multimomentum map are then (cf. eqs. (3.6)-(3.9))

\begin{equation}
(4.13) \quad \int_{\mathcal{M}} \left[ \partial_i \left( \frac{e}{N} V_1^\perp \right) + \frac{e}{N} \gamma^0 \omega^1_1 V_3^\perp \right] d^3 x_0 = 0,
\end{equation}

\begin{equation}
(4.14) \quad \int_{\mathcal{M}} \left[ \partial_i \left( \frac{e}{N} V_3^\perp \right) + \frac{e}{N} \gamma^0 \omega^3_1 V_1^\perp \right] d^3 x_0 = 0,
\end{equation}

\begin{equation}
(4.15) \quad \int_{\mathcal{M}} \left[ \partial_i \left( \frac{e}{N} V_2^\perp \right) + \frac{e}{N} \gamma^1 \omega^2_1 V_3^\perp \right] d^3 x_0 = 0,
\end{equation}

\begin{equation}
(4.16) \quad \int_{\mathcal{M}} \left[ \partial_i \Omega_{ij}^{01} \right] d^3 x_0 = 0,
\end{equation}

\begin{equation}
(4.17) \quad \int_{\mathcal{M}} \left[ e V_2^\perp \left( \gamma^0 \Omega_{ij}^1 \right) V_3^\perp + \gamma^0 \Omega_{ij}^2 V_2^\perp \right] -
\end{equation}

\begin{equation}
- \frac{2e}{N} \left[ \gamma^0 \Omega_{ij}^1 \right] V_2^\perp + \gamma^0 \Omega_{ij}^2 \bar{V}_j^3 \bar{V}_2^1 \right] \bar{\xi} d^3 x_0 = 0,
\end{equation}

\begin{equation}
(4.18) \quad \int_{\mathcal{M}} \left[ e \gamma^0 \Omega_{ij}^1 \right] V_2^\perp + \gamma^0 \Omega_{ij}^2 \bar{V}_j^3 \bar{V}_2^1 \right] \bar{\xi} d^3 x_0 = 0.
\end{equation}
On the other hand, the Euler-Lagrange equations resulting from the action (4.5) are (cf. eqs. (3.11) and (3.12))

\[ e^b_i \left( \gamma^i_{ab} \Omega_{0b} - \frac{1}{2} e^d_a e^c_h \Omega_{ad} \right) = 0 \]

and

\[ D_b \Omega^a_{ab} = 0. \]

The self-dual Einstein equations in vacuum can be thus written explicitly in the form

\[ \Gamma^{0} \equiv e^s_i \Omega_{ab}^{0} + e^s_j \Omega_{ab}^{0} = 0, \]

\[ \Gamma^{1} \equiv e^s_i \Omega_{ab}^{1} + e^s_j \Omega_{ab}^{1} = 0, \]

\[ \Gamma^{2} \equiv e^s_i \Omega_{ab}^{2} + e^s_j \Omega_{ab}^{2} = 0, \]

\[ \Gamma^{3} \equiv e^s_i \Omega_{ab}^{3} + e^s_j \Omega_{ab}^{3} = 0. \]

It is easy to show that the equations independent of time derivatives on a null hypersurface are the spatial components of eqs. (4.21) and (4.23), jointly with the equations

\[ D \Gamma^{0} \equiv D \Gamma^{0} = 0, \]

\[ D \Gamma^{2} \equiv D \Gamma^{2} = 0, \]

which are equivalent to (4.14)-(4.16), and (cf. eq. (3.7))

\[ D \Gamma^{1} \equiv D \Gamma^{1} = 0. \]

The comparison of eqs. (4.6)-(4.12) with eqs. (4.14)-(4.18) shows that eq. (4.14) corresponds to eq. (4.8), eq. (4.15) to eq. (4.10), eq. (4.16) to eq. (4.9), eq. (4.17) to eqs. (4.6) and (4.7), and eq. (4.18) to eq. (4.7).

Interestingly, the constraint equations (4.15) and (4.17), which are second order in the Hamiltonian formalism, contribute to the constraint set of the multimomentum map, and hence should be regarded as first class [7]. Indeed, if one breaks covariance, so that the internal rotation group O(3, 1) is replaced by its subgroup O(3), one can set \( \lambda^{0k} = 0, \forall k = 1, 2, 3 \). After doing this, \( \Gamma^{0} \) vanishes and eq. (4.15) reduces to an identity, while \( \Gamma^{01} \) and \( \Gamma^{12} \) remain different from zero, so that only eqs. (4.14) and (4.16) survive. As far as eq. (4.17) is concerned, one should note that, in a 3 + 1 split of space-time, Diff(M) is replaced by its subgroup Diff(J \( \cap H \)) \( \times \) Diff(\( \cap H \)). In this case, barring some mathematical rigour, one can say that the arbitrary vector field \( \xi \) admits the decomposition \( \xi^a = \xi^a + \xi^a \), where \( \xi^a \) represent the infinitesimal three-dimensional diffeomorphisms on a generic hypersurface \( \Sigma \) (for the time being, \( \Sigma \) can be either spacelike or null), and \( \xi^a \) represent the diffeomorphisms «off» this hypersurface. It should be noticed that \( \xi^a \) is a vector proportional to \( n^a = g^{ab}f_b \), where \( f = 0 \) is the equation which defines locally the hypersurface. Strictly, the normal is a field on \( \Sigma \). However, in a neighbourhood of \( \Sigma \) one can introduce a slicing of \( M \) viewed as the Cartesian product \( I \times \sigma(r) \), where \( I \) is the closed interval \( [0, \varepsilon] \) and \( \sigma \) is a
three-dimensional hypersurface obtained by moving the points of $\Sigma$ along the normal geodesics to the distance $r$, so that $s(0)$ coincides with $\Sigma$. This procedure makes it possible to extend all tensor fields defined on $\Sigma$, including the normal, to tensor fields on $M$.

In adapted coordinates, one has $\xi^i_0 = 0$ and $n^a = g^{00} \partial_0 f$. By virtue of (2.9) it follows that, on a null hypersurface, $\xi^0_\perp$ vanishes as well. This in turn implies that eq. (4.17) reduces to an identity.

These simple remarks seem to point to a deeper interpretation of our result: eqs. (4.15) and (4.17) may or may not be considered first-class constraints, depending on whether or not one breaks the original diffeomorphism group of the theory into a proper subgroup. In other words, only when $\text{Diff}(M)$ and $\text{O}(3, 1)$ are replaced by their subgroups $\text{Diff}(\mathbb{R}^3) \times \text{Diff}(\mathbb{R})$ and $\text{O}(3)$, the constraints (4.15) and (4.17) become second class and hence do not contribute to the multimomentum map.

5. - Concluding remarks

This paper has considered the application of the multimomentum-map technique to study general relativity as a constrained system on null hypersurfaces. Its contribution lies in relating different formalisms for such a constraint analysis. We have found that, on a null hypersurface, the multimomentum map provides just a subset of the full set of constraints, regarded as those particular Euler-Lagrange equations which are not of evolutionary type.

Although the multimomentum map is expected to yield only the secondary first-class constraints [8], we have found that some of the constraints which are second class in the Hamiltonian formalism occur also in the Lagrangian multimomentum map. This leads to inequivalent formalisms. Such inequivalence can be interpreted by observing that our analysis remains covariant in that it deals with the full diffeomorphism group of space-time, say $\text{Diff}(M)$, jointly with the internal rotation group $\text{O}(3, 1)$. Hence one incorporates some constraints which are instead ruled out if one breaks covariance, which amounts to taking subgroups of the ones just mentioned (see sect. 4). The remaining (second-class) constraints have been found just by checking which Euler-Lagrange equations are not of evolutionary type. This property suggests, perhaps, that second-class constraints can be treated by introducing some modifications in the construction of the multimomentum map. The work in ref. [7] shows that a suitable definition of «momentum map» may be introduced, so as to incorporate the analysis of primary first-class constraints as well. Of course, this is no longer a Lagrangian analysis [11], but appears to be an important issue for further research.

Another relevant problem lies in the constraint analysis on double null hypersurfaces. These consist of two null hypersurfaces, intersecting each other in a spacelike two-surface [5, 12]. On a double null hypersurface, both the integrations $d^3x_0$ and $d^3x_1$ (cf. eq. (3.1)) survive in the multimomentum-map equations.

The multisymplectic framework appears to have very interesting features both in general relativity and in other field theories [7], but our paper shows that there is still an unsatisfactory state of affairs in this formalism because of the lack of a systematic algorithm to generate all constraints of the theory, since they have been found just by inspection of the field equations. Perhaps one needs a suitable version of covariant Hamiltonian formalism for constrained systems (cf. refs. [13, 14]). A proper under-
standing of all the above issues will show, presumably, whether or not the multi-
momentum-map formalism offers substantial advantages with respect to the well-
established Hamiltonian techniques [1-4, 6].

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R E F E R E N C E S