Gravity without the metric, torsion and the cosmological-constant problem

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Summary. — In the gravity without the metric formulation of Capovilla, Jacobson and Dell, we can introduce a \( \theta \)-term which corresponds to torsion in the SL(2, C) gauge-theoretical framework. Torsion induces a topological phase to the matter wave function in the Lorentzian sector and in a local Lorentz frame, where it appears momentarily to be at rest, this phase may be associated with wormhole charge when in the Euclidean sector torsion induces the formation of instanton-anti-instanton pairs in disjoint spaces leading to the formation of wormholes. This provides a link between Coleman's solution of the cosmological-constant problem and the acquirement of the topological phase to the matter wave function in the Lorentzian sector, which leads to the vanishing of the vacuum energy density. This circumvents the problem associated with wormholes and Euclidean cosmology.

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Recently Capovilla, Jacobson and Dell [1] have formulated a theory of gravitation without metric which considers a Lagrangian formulation of Ashtekar's [2] theory in which the metric or the triad has been completely eliminated in favour of connection. In 3 + 1 dimensions, the action in terms of Ashtekar's variable can be written as

\[
S = \int \mathcal{A}_a E^a - N^a \mathcal{H} - N^a \mathcal{H}_a - A_\theta \mathcal{J}_1,
\]

\[
\mathcal{H} = \frac{1}{2} i f_{ijk} E^i E^j F_{abk} = \frac{1}{2} i \varepsilon_{abc} f_{ijk} E^i E^j B_{k}^c,
\]

\[
\mathcal{H}_a = E^b F_{abk} = \varepsilon_{abc} E^i B_{k}^c,
\]

\[
\mathcal{J}_1 = D_a E^a = \partial_a E^a + i f_{ijk} A_{aj} E^k.
\]

Here \( a, b, c \) are spatial indices, \( i, j, k \) are SO(3) indices, \( F_{abk} \) is an SO(3) curvature, \( B_{k}^c = (1/2) \varepsilon_{abc} F_{bci} \) is the corresponding magnetic field; Capovilla, Jacobson and Dell have

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considered the Lagrangian formulation of the above theory. The CJ D action is

\begin{equation}
S^{\text{CJD}} = \frac{1}{8} \int \eta (\Omega_{ij} \Omega_{ji} + a \Omega_{ii} \Omega_{jj}),
\end{equation}

where

\begin{equation}
\Omega_{ij} = \epsilon^{\alpha \beta \gamma \delta} F_{\alpha \beta i} F_{\gamma \delta j}.
\end{equation}

Here, \( \alpha, \beta \) are space-time indices, the Lagrange multiplier \( \eta \) is a scalar density of weight one and \( F_{\alpha \beta i} \) is an SO(3) field strength. They showed that a \( 3 + 1 \) decomposition of this action yields Ashtekar's action directly, provided that the parameter \( a = -1/2 \) and the determinant of the magnetic field \( B^{\alpha} \) is non-zero. When the CJ D action is varied with respect to the Lagrange multiplier \( \eta \), it is actually the Hamiltonian constraint in disguise,

\begin{equation}
\psi = \Omega_{ij} \Omega_{ji} - \frac{1}{2} \Omega_{ii} \Omega_{jj} = i (2 \eta^2 \text{det} B)^{-1} \mathcal{H}.
\end{equation}

Let us now introduce the matrix \( \psi_{ij} \) defined by

\begin{equation}
E_i^a = \psi_{ij} B_j^a,
\end{equation}

where we note that such a matrix always exists provided that the magnetic field is non-degenerate.

Bengtsson and Peldan [3] have shown that, when a canonical transformation

\begin{equation}
A_{ai} \rightarrow A_{ai}, \quad E_i^a \rightarrow E_i^a - \theta B_i^a
\end{equation}

is given, the expression for the Hamiltonian constraint changes though the remaining constraints are unaffected. This corresponds precisely to the addition of a CP-violating \( \theta \)-term to the CJ D Lagrangian when the new action is given by

\begin{equation}
S = \frac{1}{8} \int \theta \Omega_{ii} + \eta \left( \Omega_{ij} \Omega_{ji} - \frac{1}{2} \Omega_{ii} \Omega_{jj} \right).
\end{equation}

It has been shown that, when the gauge group is taken to be SL(2, C) where \( i, j \) correspond to the SL(2, C) indices, this \( \theta \)-term now corresponds to the Lagrangian

\begin{equation}
L = - \frac{1}{4} \theta \text{Tr} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}
\end{equation}

with

\begin{equation}
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],
\end{equation}

where \( A_\mu = A_\mu \cdot g \), \( F_{\mu \nu} = F_{\mu \nu} \cdot g \), \( g^1, g^2, g^3 \) being the SL(2, C) generators. This gives rise to the current

\begin{equation}
j_\mu^a = \epsilon^{\mu \nu \rho \sigma} A_\nu \times F_{\rho \sigma} = \epsilon^{\mu \nu \rho \sigma} \partial_\nu F_{\rho \sigma},
\end{equation}
which in turn is responsible for torsion [4]. Indeed, if we take

\[ \mathbf{A}_\nu \times \mathbf{F}_{\alpha \beta} = \kappa^2 S_{\nu\alpha\beta} \mathbf{n}, \]

\( \kappa \) being the Planck length, \( \mathbf{n} \) being the unit vector, the coupling \( j^\mu \cdot j^\nu \) yields the action

\[ S = -\frac{4}{\kappa^2} \int S_{\nu\alpha\beta} S_{\mu\gamma\delta} d^4 x \]

leading to torsion. This term is related with the chiral anomaly which arises when chiral currents interact with a gauge field [5]. When we describe a matter field in the geometry associated with the Lagrangian (8), we note that, in the background of the \( \text{SL}(2, \mathbb{C}) \) gauge fields, the Lagrangian for a Dirac spinor field may be written as

\[ L = -\bar{\psi} \gamma^\mu D_\mu \psi - \frac{1}{4} \text{Tr} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \]

where \( D_\mu \) is the \( \text{SL}(2, \mathbb{C}) \) gauge covariant derivative defined by \( D_\mu = \partial_\mu - igA_\mu \), where \( g \) is some coupling strength. Splitting the Dirac massless spinor in chiral forms and identifying the internal helicity \( +1/2 \) \((-1/2)\) with left (right) chirality, we have the following three conservation laws [6]:

\[ \left\{ \begin{array}{l}
\partial_\mu \left[ \frac{1}{2} ( -ig\bar{\psi}_R \gamma^\mu \psi_R + j^\mu) \right] = 0, \\
\partial_\mu \left[ \frac{1}{2} ( -ig\bar{\psi}_L \gamma^\mu \psi_L + ig\bar{\psi}_R \gamma^\mu \psi_R + j^\mu) \right] = 0, \\
\partial_\mu \left[ \frac{1}{2} ( -ig\bar{\psi}_L \gamma^\mu \psi_L + j^\mu) \right] = 0.
\end{array} \right. \]

These three equations represent a consistent set of equations, if we choose

\[ j^1 = -\frac{1}{2} j^2, \quad j^2 = +\frac{1}{2} j^3. \]

Substituting (15) into (14) we can get

\[ \partial_\mu (\bar{\psi} \gamma^\mu \gamma^5 \psi) = \partial_\mu j^\mu = -2 \partial_\mu j^\mu. \]

Now the gauge field Lagrangian is related to the Pontryagin density

\[ P = -\frac{1}{16\pi^2} \epsilon^{\alpha\nu\sigma\beta} \text{Tr} F_{\alpha\beta} F_{\gamma\delta} = \partial_\mu \Omega^\mu, \]

where

\[ \Omega^\mu = -\frac{1}{16\pi^2} \epsilon^{\nu\mu\alpha\beta} \text{Tr} \left[ A_\nu F_{\alpha\beta} - \frac{2}{3} (A_\nu A_\alpha A_\beta) \right] \]
is the Chern-Simons secondary characteristic class. The Pontryagin index $q = \int P \, d^4x$ is a topological invariant. From the relation (16), it is noted that $\Omega_{\mu}$ is associated with $j_{\mu}^2$ and the Pontryagin index can be expressed as $[7]$

$$q = \int j_\delta \, d^4x = \int \partial_\mu j_\mu^2 \, d^4x. \tag{19}$$

From the relation (16) we can write

$$j_\mu^2 = - \frac{1}{2} (j_\mu^5 + j_\mu^\gamma),$$

where $j_\mu^\gamma$ is any arbitrary vector current which is conserved. We can take the particular solution

$$j_\mu^2 = - \frac{1}{2} j_\mu^5 = - \frac{1}{2} \psi \gamma_\mu \gamma^5 \psi, \tag{20}$$

$$\partial_\mu j_\mu^2 = - \text{im} \, \bar{\psi} \gamma^5 \psi. \tag{21}$$

Again,

$$\partial_\mu j_\mu^2 l = - 2 \text{ im} \, \gamma_\mu j_\mu^2, \tag{22}$$

where $l$ is the identity matrix. So

$$-4m^2 \int j_\mu^2 j_\mu^2 \, d^4x = \int (\partial_\mu j_\mu^2)(\partial_\mu j_\mu^2) \, d^4x = qC, \tag{23}$$

where $C$ is any arbitrary constant. Thus the torsion term given by $\int j_\mu^2 j_\mu^2 \, d^4x$ actually corresponds to the Pontryagin index $q$ and the net effect of the $\theta$-term in the action is just to introduce the torsion term $[7]$. It is noted that the $\theta$-term which corresponds to torsion is a topological term. If we take the Hermitian representation of the $\text{SL}(2, \mathbb{C})$ group structure, we can take the compact group $\text{SU}(2)$ as the group manifold. Now we consider a compact region within which $\bar{\psi} j_\mu^2 \neq 0$, but outside which $\partial_\mu j_\mu^2 = 0$. This implies that outside the compact space we have only the $\theta$-term in the action, as the Einstein-Hilbert action for pure gravity will be vanishing here. This follows from the fact that there cannot be any matter field in this region, as any spinorial matter when written in chiral form demands $\bar{\psi} j_\mu^2 \neq 0$, as discussed in the previous section. Thus the boundary of the compact space may be taken to be the nucleation point. It may be recalled here that in a recent paper $[6]$ we have shown that the chiral anomaly $(\partial_\mu j_\mu^2 \neq 0)$ may be taken to be responsible for the origin of mass and in the region where $\partial_\mu j_\mu^2 = 0$ there is no nucleation.

On the nucleation boundary, the torsion term effectively corresponds to the cosmological constant. Indeed from the relation (10)

$$j_\mu^{(2)} = e^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma}^{(2)}$$

and noting the antisymmetric nature of $F_{\lambda\sigma}$, we can write

$$j_\mu^{(2)} = e^{\mu\nu\lambda\sigma} e_{\nu\lambda\sigma} C(x), \tag{24}$$

where $C(x)$ is a scalar function. Now from the relation $\partial_\mu j_\mu^2 = 0$ we find that $C(x)$ is
constant. So the torsion term $j^2$ gives rise to the constant $C^2$ which now appears as a cosmological constant.

Now in the region where $\partial_{\mu} j^2 = 0$ everywhere outside a compact space, we have only the topological Lagrangian

$$L = \theta \text{Tr}^* F^{\mu\nu} F_{\mu\nu}. \quad (25)$$

Indeed there is no massive matter in this region characterized by the absence of nucleation and hence the Einstein part corresponding to the curvature vanishes. Thus the Lagrangian here just corresponds to the cosmological term

$$L = \Lambda \sqrt{g} \quad (26)$$

which may be renormalized to the Lagrangian

$$L = 0, \quad (27)$$

where we take $F_{\mu\nu} = 0$ corresponding to the pure gauge condition $A_{\mu} = U^{-1} \partial_{\mu} U$. This Lagrangian has more symmetries than the usual diffeomorphism invariances. Evidently, general covariance is unbroken here and we have broken symmetry as the nucleation starts at the boundary.

It is evident from the CJD formalism that it provides a link between covariant and canonical quantization of gravity. Indeed, in recent papers [8, 9] it has been pointed out that we may achieve the Wheeler-DeWitt equation when we take the limiting case of torsion tending to zero. It may be noted that the Hamiltonian constraint gets modified by the $\theta$-term though other constraints remain unaffected and so the Wheeler-DeWitt equation effectively demands the vanishing of the $\theta$-term representing torsion.

From our above discussion, we have noted that the $\theta$-term which corresponds to torsion envisages a compact region where $\partial_{\mu} j^2 \neq 0$, but outside it, where the torsion term vanishes, we have $\partial_{\mu} j^2 = 0$. Now from eqs. (8) and (10), we note that this picture may be visualized by taking into consideration that inside the compact region $F_{\mu\nu} \neq 0$ but outside it we have $F_{\mu\nu} = 0$. Taking the Hermitian representation, we can consider the group space of gauge fields as $SU(2)$. Again in the Euclidean space-time $R^4$ we can take the boundary as $S^3$. Hence topologically non-trivial solutions to the $SU(2)$ gauge field equations are possible when we consider a mapping from the group space $S^3$ ($SU(2) = S^3$) onto the 3-sphere $S^3$. This leads to instantons. Indeed, we have noted that the Pontryagin index is given by the $\theta$-term through the relation

$$q = -\frac{1}{16\pi^2} \int \text{Tr}^* F^{\mu\nu} F_{\mu\nu} d^4x = \int \partial_{\mu} \Omega^{\mu} d^4x \quad (28)$$

which is again related to the torsion generating current through eq. (19). This index may be taken to be the winding number of the mapping from the group space $S^3$ to the 3-sphere $S^3$ which is the boundary of the Euclidean space-time and can be expressed as

$$q = \frac{1}{24\pi^2} \int d^4s \epsilon^{\mu\nu\kappa\lambda} \text{Tr} \{ (U^{-1} \partial_{\nu} U)(U^{-1} \partial_{\kappa} U)(U^{-1} \partial_{\lambda} U) \}, \quad (29)$$

where on the nucleation boundary we take $F_{\mu\nu} = 0$ and we have the pure gauge condition $A_{\mu} \to U^{-1} \partial_{\mu} U$. This helps us to identify the compact region where $F_{\mu\nu} \neq 0$
as an instanton. Now we note from eqs. (14) and (15) that $\pm j_{\mu}^2$ effectively corresponds to two opposite chiralities of the matter field, and so we can visualize that in the Euclidean sector the torsion associated with the $j_{\mu}^2$ current gives rise to instanton-anti-instanton pairs at some disjoint regions of space-time. So by joining the periphery of such pairs wormhole handles are formed and the compact regions may be interpreted as wormhole mouths. In particular, we can think of such compact regions as closed universes which are allowed to branch off or join onto our asymptotic flat region of space-time. The two compact 4-spaces in the Euclidean sector are connected to each other with Planck-sized wormholes. Each wormhole can be regarded as the creation and subsequent destruction of a baby universe which may or may not appear as part of the boundary of the 4-manifold.

This analysis now helps us to understand Coleman’s formalism for the cosmological-constant problem which takes into account the effect of creating and destroying arbitrary numbers of baby universes. In fact, taking $|B\rangle$ as the normalized baby universe state which is expanded in eigenstates of the operators $(a_i + a_i^\dagger)$, where $a_i (a_i^\dagger)$ is the annihilation and creation operator for a baby universe such that

$$
|B\rangle = \int f_B(\alpha) \prod_i d\alpha_i |\alpha\rangle,
$$

and $f_B(\alpha)$ depending on the boundary condition, the effective action is changed to

$$
S_{\alpha} = S + \sum_i \alpha_i \int O_i(x) \, d^4x,
$$

where $O_i(x)$ is the corresponding local operator. Coleman observed that on a scale much larger than the wormhole scale those manifolds that appear disconnected can be interpreted as connected by wormholes. The path integral then suggests an infinite peak at $\lambda_{\text{eff}} \to 0$, where $\lambda_{\text{eff}}$ is the cosmological constant [10]. As has been pointed out by several authors [11], the Euclidean formalism has its pathological defect as we cannot have the quantum-mechanical probability interpretation. And it appears that in the Lorentzian metric, the whole argument becomes invalid. However, we note here that in our formalism of gravity without the metric including a $\theta$-term, wormholes appear in the Euclidean sector when outside a compact region we have vanishing torsion $(\partial_{\nu} j_{\mu}^2 = 0)$. Indeed on the boundary of the compact region inside which $F_{\mu\nu} \neq 0$ but outside which we have $F_{\mu\nu} = 0$, we find that the Hamiltonian constraint and hence the Wheeler-De Witt equation arise when we take the limiting case of torsion tending to zero. It is noted that the $\theta$-term is $T$-violating, indicating that it suggests an arrow of time. So on the surface where we take it that the torsion term tends to zero the sense of this arrow of time is suppressed. This can be interpreted in two ways:

1) In the analysis of the Wheeler-De Witt equation there should not be any reference to extrinsic time in the Heraclitian sense. That is, there is no time, no before, no after.

2) We can think that we have the a priori information that the arrow of time was there and on the constant-time surface we have the limiting effect but in all
mathematical and interpretative formalisms we should bear in mind this arrow of time.

In the first case we can resort to Euclidean space-time and Coleman’s formalism may be considered to be valid, though we have difficulty with the probability interpretation. However, in the second case we should take into account Lorentzian metric. As Vilenkin [12] has pointed out, the quantum state of the universe can in this case be taken to arise in an analogous way of quantum tunneling through a potential barrier where the universe spontaneously nucleates in a de Sitter space. The boundary condition is now given by the fact that we have only outgoing waves. It has been already pointed out [13, 14] that the $\theta$-term associated with the non-Abelian gauge field has the geometrical implication that it corresponds to a vortex line. The torsion term may be viewed as arising out of this vortex line attached to a space-time point and this picture of microlocal space-time is effectively associated with the spinor structure [7]. In such a space-time, the field function will be of the form $\psi(x^\alpha, \xi^\mu)$ which can be treated to describe a particle moving in an anisotropic space with the direction vector (vortex line) $\xi^\mu$ attached to the space-time point $x^\alpha$. In this case, the wave function should take into account the polar coordinates $r, \theta, \phi$ along with the angle $\chi$ depicting the rotational orientation around the direction vector $\xi^\mu$. The eigenvalue $\mu$ of the operator $i(\partial/\partial x^\mu)$ just corresponds to internal helicity. In three space dimensions, these three angles have their correspondence in an axisymmetric system where the anisotropy is introduced along a particular direction. The components of the linear momentum now satisfy a commutation relation of the form

\begin{equation}
[p_i, p_j] = i\mu e_{ijk}(x^k/r^3)
\end{equation}

and the angular momentum is given by

\begin{equation}
J = r \times p - \mu r.
\end{equation}

This is similar to the system of a charged particle moving in the field of a magnetic monopole. Here $\mu$ can take the values 0, $\pm 1/2$, $\pm 1$, $\pm 3/2$, ... and hence in this space a particle can move with $l = 1/2$. It may be added that the spherical harmonics $Y_l^m$ incorporating the term $\mu$ have been extensively studied by Fierz [15] and Hurst [16].

In a gauge-field-theoretic framework we can consider a loop integral $\text{Tr} P \exp[iA_\mu dx^\mu]$ corresponding to the holonomy and this is associated with the topological phase (Berry phase). In fact the Berry phase is linked up with chiral anomaly which is again associated with the $\theta$-term. The charge corresponding to the gauge field part is

\begin{equation}
q = \int j_\mu^2 d^3x = \int \varepsilon^{ijk} d\sigma_i F^{(2)}_{jk}(i, j, k = 1, 2, 3).
\end{equation}

Visualizing $F^{(2)}_{jk}$ to be the magnetic-field–like components for the vector potential, we see that $q$ is actually associated with the magnetic-pole strength for the corresponding field distribution. This charge is effectively given by the Pontryagin index in this formalism and is related to chiral anomaly through the relations (16) and (19):

\begin{equation}
q = \int \partial_\mu j_\mu^2 d^4x = -\frac{1}{2} \int \partial_\mu j_\mu^2 d^4x.
\end{equation}
The Berry phase \([17]\) \(e^{i\phi_B}\) generated in a closed parameter space is given by \([18]\)

\[
\phi_B = 2\pi \mu
\]

with \(q = 2\mu\). Thus we find that the torsion term gives rise to a non-zero Berry phase which is associated with the change in chirality over a closed path. It may be added that the Aharonov-Bohm phase which is related to a multiply connected nature of space-time generated by a solenoid is a special case of Berry phase.

The acquirement of this phase by the wave function corresponding to matter field suggests that this can move with \(l = 1/2\) as the quantity \(\mu\) behaves as a magnetic-pole strength. In fact the inherent anisotropic nature of space-time helps us to associate \(\mu\) with the change in chirality associated with the angle \(\chi\) depicting the rotation around the vortex line where we have the eigenvalue relation \(i(\partial/\partial\chi)\psi = \mu\psi\), and when the angle \(\chi\) is changed over the closed path \(0 \leq \chi \leq 2\pi\) for one such complete rotation, the wave function will acquire the phase \(e^{i\mu\chi}\). The association of \(\mu\) with the spherical harmonics \(Y^m_\mu\) incorporates the fact that the particle is now allowed to move with \(l = 1/2\) in such a space. So when we take it that the basic matter ingredient is a fermion, we find that the vacuum energy density in such a space vanishes as in the ground state we will have the relation

\[
E = \left(1 - \frac{1}{2}\right)\hbar\omega = 0 \quad \text{for} \quad l = \frac{1}{2}.
\]

This analysis suggests that in a local Lorentz frame where it appears momentarily to be at rest, this phase factor may be associated with the wormhole charge where we consider that these non-contractible loops lead to the generation of wormhole handles connecting the periphery of two such loops. The loops appear to be wormhole mouths. In a gauge-theoretical framework we note that these loop integrals are gauge-invariant objects and this gauge invariance allows us to interpret the topological phase without any reference to extrinsic time. Moreover, the matter eigenstates do not have dynamical phase as these presuppose an a priori extrinsic time. This helps us to identify Coleman’s formalism of the cosmological-constant problem in the Euclidean sector with the vanishing of the vacuum energy density in the Lorentzian sector. In fact, the peak in the wave function in the Euclidean sector is found to be associated with the acquirement of the topological phase by the wave function of a matter field in the Lorentzian sector, which makes the vacuum energy density vanish. In this way we can have an interpretation of Coleman’s formalism in the Lorentzian metric formulation. Our above analysis suggests that we can have a solution of the cosmological-constant problem in the Lorentzian sector when we find that torsion effectively induces a topological phase to the matter wave function, which makes the vacuum energy density vanish. In the Euclidean sector this may be associated with wormhole dynamics when we have torsion-induced wormholes formed by joining instanton-anti-instanton pairs in a disjoint space-time. This provides a link between Coleman’s formalism of the cosmological-constant problem in the Euclidean sector and a corresponding effect of the vanishing of the vacuum energy density through the effect of this topological phase in the Lorentzian sector. This helps us to circumvent the objections raised by Coleman himself regarding his formalism as stated in the following words \([10]\): “It rests on wormhole dynamics and the Euclidean formulation of quantum gravity. Thus it is doubly built on sand. Wormholes may not exist or if may do exist their effects may be overwhelmed by those of some more exotic configurations. Likewise, the Euclidean
formulation of gravity is not a subject with firm foundation and clear rules of procedure, indeed it is more like a trackless swamp".

Finally, we may point out that in the gravity without the metric formalism incorporating the $\theta$-term, we find a link between the covariant quantization of gravity and the canonical quantization when the Wheeler-DeWitt equation arises as the torsion term tends to be vanishing. The $T$-violating $\theta$-term associated with torsion suggests that implicitly we should bear in mind the sense of the arrow of time which helps us to formulate the system in the Lorentzian sector. When we forget about time, we can go to the Euclidean sector. The link between Coleman's formalism of the cosmological-constant problem in the Euclidean sector and the topological-phase formalism in the Lorentzian sector may be viewed in an analogous way.

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