Metric-affine gravitation theory and superpotentials (*)

G. GIACCHETTA, L. MANGIAROTTI (**) and A. SALTARELLI
Dipartimento di Matematica e Fisica, Università di Camerino
62032 Camerino (MC), Italy

(ricevuto il 17 Luglio 1996; approvato il 30 Agosto 1996)

Summary. — We consider a metric-affine theory of gravity in which the dynamical fields are the Lorentzian metrics and the non-symmetric linear connections on the world manifold $X$. Working with a Lagrangian density which is invariant under general covariant transformations and using standard tools of the calculus of variations, we study the corresponding currents. We find that the superpotential takes a nice form involving the torsion of the linear connection in a simple way and generalizing the well-known Komar superpotential. A feature of our approach is the use of the Poincaré-Cartan form in relation to the first variational formula of the calculus of variations.

PACS 02.40 – Geometry, differential geometry and topology.
PACS 04.20.Fy – Canonical formalism, Lagrangians, and variational principles.

1. – Introduction

As is well known, generally covariant field theories lead to strong conservation laws. This means that when the field equations are satisfied the current $J^\lambda$ associated with the energy-momentum tensor can be written as

\[ J^\lambda = d_\mu U^{\lambda\mu}, \]

where $d_\mu$ is the formal (total) derivative and the skew-symmetric tensor density $U^{\lambda\mu}$ is called the superpotential. For example, in the purely metric Einstein's gravitation theory, the superpotential corresponding to the Einstein-Hilbert Lagrangian density $L_{EH} = \sqrt{-g} R$ takes the form

\[ U^{\lambda\mu} = \sqrt{-g} (\nabla_\mu u^\lambda - \nabla^\lambda u_\mu g^{\mu\nu}). \]

This is the well-known Komar superpotential [1] associated with the vector field $u^\lambda$ on

(*) The authors of this paper have agreed to not receive the proofs for correction.
(**) E-mail: mangiarotti@camvax.cineca.it
the world manifold \( X \). Here \( \nabla_\alpha \) denotes the covariant derivative with respect to the Levi-Civita connection.

In this paper we are concerned with the metric-affine theory of gravity. Its configuration space is coordinatized by \((x^l, g^{lm}, k^a_{\beta\lambda})\), where \( x^l \) are local coordinates on the world manifold \( X \), while \( g^{lm} \) and \( k^a_{\beta\lambda} \) are the coefficients of a Lorentzian metric and a non-symmetric linear connection on \( X \), respectively. We consider a Lagrangian density \( L(x^l, g^{lm}, k^a_{\beta\lambda}, k^a_{\beta\lambda}, m^a) \), where as usual \( k^a_{\beta\lambda}, m^a \) are the momenta corresponding to the connection parameters. The quantity (3) generalizes the Komar superpotential (2). Indeed, in the case of the Einstein-Hilbert Lagrangian density, using the Palatini variables, we have

\[
U^{\alpha\mu} = \pi^{\alpha}_{\beta\mu}(\nabla_\beta u^{\alpha} + T^{\alpha}_{\alpha\beta\mu} u^{\alpha}).
\]

Now \( \nabla_\beta \) is the covariant derivative with respect to the linear connection \( k^a_{\beta\lambda} \), \( T^{\alpha}_{\alpha\beta\mu} = k^a_{\alpha\beta\mu} - k^a_{\alpha\lambda\mu} \) are the corresponding torsion parameters, and

\[
\pi^{\alpha}_{\beta\mu} = \partial L / \partial k^a_{\beta\lambda, \mu}
\]

are the momenta corresponding to the connection parameters. The quantity (3) produces the Komar superpotential (2).

There are different methods to discover differential conservation laws in Lagrangian field theories. In this work we are concerned with the so-called symmetry method. To illustrate the generality of the situation, in the following section we first consider some basic concepts of the Lagrangian formalism. Then we derive our main tool, the variational formula. Section 3 contains the characterization of generally covariant Lagrangians (in the metric-affine gravitation theory) which is used, in the final section, to derive our results concerning currents and superpotentials.

2. - Lagrangian formalism

In this section we consider some basic tools of the Lagrangian formalism [5]. In particular, we use the Poincaré-Cartan form to derive, in a new way, both the Euler-Lagrange operator and the (canonical) energy-momentum tensor. These objects will be the ingredients of our basic formula, namely the variational formula.

2.1. General considerations. - Let us consider a (first-order) field theory over a world manifold \( X \) locally parametrized by the coordinates \( x^\lambda, 1 \leq \lambda \leq m = \dim X \), whose classical fields are denoted by \( y^i, 1 \leq i \leq l \). As usual, we will denote by \( \partial_\lambda = \partial / \partial x^\lambda, \partial_i = \partial / \partial y^i, y^i_\lambda = \partial_i y^i \) (the field derivatives) and \( y^i_\lambda = \partial / \partial y^i \).

A basic object is the canonical Liouville form [4]

\[
\Theta = p^i_\lambda \theta^i \wedge \omega_\lambda,
\]

where \( p^i_\lambda \) are the field momenta associated with the fields \( y^i \) and

\[
\theta^i = dy^i - y^i_\lambda dx^\lambda, \quad \omega_\lambda = \partial_\lambda \omega, \quad \omega = dx^1 \wedge \ldots \wedge dx^m.
\]
Here the symbols $\wedge$ and $\langle$ stand for the wedge product and the inner product, respectively.

Later we will call configuration space, velocity phase space and momentum phase space of the field system the manifolds locally coordinatized by $(x^i, y^j)$, $(x^i, y^j, y^l)$, and $(x^i, y^j, p_i)$, respectively.

As is well known, the starting point of the Lagrangian formalism is a Lagrangian density $L = L(x^i, y^j, y^l)$. As in analytical mechanics, we can use the Legendre map

$$
p_l = \frac{\partial L}{\partial y^l} \tag{8}
$$

then the Poincaré-Cartan form associated with the Lagrangian density $L$ can be defined as

$$
\mathcal{H}_L = L \, \omega^i \wedge \omega_i - H \, \omega, \quad H = y^l \frac{\partial L}{\partial y^l} - L. \tag{9}
$$

Taking the vertical exterior differential [4] of $\Theta_L$, we get

$$
\begin{cases}
\Omega_L = d_L \Theta_L, \\
\Omega_L = (\partial_L \frac{\partial L}{\partial y^l} \omega^i \wedge \omega_i) \wedge \omega l, \\
\theta^i = dy^i - y^j dx^j, \quad \theta^l = dy^l - y^j dx^j,
\end{cases} \tag{10}
$$

where $y^j dx^j = \partial_L \frac{\partial L}{\partial y^l} \omega^i$. Then the Euler-Lagrange operator associated with the Lagrangian density $L$ can be defined as

$$
\mathcal{E}_L = d \mathcal{H}_L - \Omega_L, \quad \mathcal{E}_L = (\partial_L \omega^i \wedge \omega_i) \wedge d_L, \quad d_L = \partial_L + y^j \partial_j + y^j \partial_j, \tag{11}
$$

where $d$ is the exterior differential.

22. The variational formula. – A slight modification of the Poincaré-Cartan form $\mathcal{H}_L$ as defined in (9) leads to the (canonical) energy-momentum tensor. First of all, note that the Liouville form $\Theta_L$ (associated with $L$) given by (8) can be written in the following way:

$$
\Theta_L = \frac{\partial L}{\partial y^l} \omega^i \wedge \omega_i. \tag{12}
$$

Then using the canonical injection

$$
i : \omega^i \mapsto \omega^i \wedge dx^i, \tag{13}
$$
we define the (canonical) energy-momentum tensor associated with the Lagrangian density \( \mathcal{L} \) as

\[
T = \left( i \mathcal{L} + \Theta \right),
\]

\[
T = \omega_\lambda \otimes \left[ (\delta_\lambda^\mu \mathcal{L} - y_\mu \partial_\mu \mathcal{L}) \, dx^\lambda + \partial_\lambda \mathcal{L} \, dy^\lambda \right].
\]

Of course, \( T \) is a true tensor globally defined if \( \mathcal{L} \) is so.

To understand the meaning of (14) let

(15)

\[ u = u^\lambda \partial_\lambda + u^i \partial_i, \]

be a vector field on the configuration space of the field system, where \( u^\lambda \) are local functions on \( X \) while \( u^i \) can also depend on the field variables. Then the corresponding vector field on the velocity phase space reads [2]

\[
\begin{align*}
\pi &= u^\lambda \partial_\lambda + u^i \partial_i + u^\lambda_i \partial_i, \\
u^\lambda_i &= d_\lambda u^i - y_\mu \partial_\mu u^i = \partial_\lambda u^i + y_\mu \partial_\mu u^i - y_\mu \partial_\mu u^\lambda.
\end{align*}
\]

Note that \( u \) is vertical if \( u^\lambda = 0 \). In general, the vertical part \( u_V \) of \( u \) can be defined as follows:

(17)

\[ u_V = \left( u^i - y_\mu u^\lambda \right) \partial_i. \]

Note that \( u = u_V \) iff \( u^\lambda = 0 \).

Given a vector field \( u \), the current \( J \) associated with it is defined by the contraction of \( u \) with the energy-momentum tensor \( T \) given by (14), i.e.

\[
\begin{align*}
J &= u^i T^i, \\
J &= J^i = u^i, J^i + y_\mu \partial_\mu \partial_\lambda u^i = \partial_\lambda u^i + y_\mu \partial_\mu u^\lambda + y_\mu \partial_\mu u^i.
\end{align*}
\]

Equivalently, using the Poincaré-Cartan form \( \mathcal{A}_x \) given by (9) and the well-known contact form

(19)

\[ \lambda = dx^\lambda \otimes \left( \partial_\lambda + y_\mu \partial_i \right), \]

we see that the current \( J \) can also be written as

(20)

\[ J = \lambda^* \left( u^i \mathcal{A}_x \right), \]

where the projection \( \lambda^* \), given by

(21)

\[ \lambda^* \, dx^\lambda = dx^\lambda, \quad \lambda^* \, dy^i = y_\mu \, dx^\lambda, \]

is the transpose of the injection \( \lambda \) in (19). The current \( J \) can be integrated over appropriate \((m - 1)\)-dimensional hypersurfaces of the world manifold \( X \) to produce global conserved quantities [6].

Lemma 2.1 (The variational formula). Let \( \mathcal{L} \) be a Lagrangian density and \( u \) be a vector field on the configuration space as in (15). Then the following identity holds:

(22)

\[ L_\sigma \mathcal{L} = u_V J \mathcal{A}_x + d_\sigma J. \]
On the left side of this identity we have the Lie derivative of $\mathcal{L}$ with respect to $\mathfrak{u}$. On the right side we have two terms: the first is the contraction of the Euler-Lagrange operator $\mathcal{L}^\ast$ with the vertical part $\mathfrak{u}_V$ of $\mathfrak{u}$, while the second is the horizontal (formal) derivative of the current $J$, i.e.

$$d_H J = d_J J^\ast \omega.$$  

Proof. It is a direct check which follows from (16), (11), (17) and (20).

Remark 2.2 (Noether’s theorem). – Suppose that the vector field $\mathfrak{u}$ is a symmetry of the Lagrangian density $\mathcal{L}$, i.e. $L_{\mathfrak{u}} \mathcal{L} = 0$. Then $d_H J = 0$ on the solutions of the field equations.

3. Metric-affine gravitation theory

The total configuration space of the metric-affine gravitation theory is parametrized by $(x^l, g_{\alpha \beta}, k_{\alpha \beta l})$, where $g_{\alpha \beta}$ and $k_{\alpha \beta l}$ are the coefficients of a Lorentzian metric and the connection parameters of a non-symmetric linear connection on the world manifold $X$, respectively (of course, $X$ is assumed to satisfy the well-known topological conditions [7] to be provided with a Lorentzian metric). In this section, first we characterize the generally covariant Lagrangian densities related to the configuration space of the metric-affine gravitation theory, then, in particular, we consider those whose dependence on the linear connections on $X$ is through their curvature tensor. Moreover, some interesting properties of the curvature tensor are discussed.

3.1. Generally covariant Lagrangians. – It is easily seen that a vector field

$$\mathfrak{u} = u^\lambda \partial_\lambda$$
on the world manifold $X$ ($u^\lambda$ are local functions on $X$) induces vector fields $\mathfrak{u}_M$ and $\mathfrak{u}_C$ on the configuration spaces of the Lorentzian metrics [3] and the linear connections. The field $\mathfrak{u}_M$ is given by

$$\mathfrak{u}_M = u^\lambda \partial_\lambda + (g^\alpha \beta u^\mu \partial_\lambda g_{\beta \mu} + g^{\alpha \beta} \partial_\lambda u^\alpha) \partial_\alpha \beta$$

(\partial_{\alpha \beta} = \partial / \partial g^{\alpha \beta}). Now from the well-known affine transformation rule of the connection parameters under a change of coordinates, i.e.

$$x_\mu = x^\mu (\mathfrak{x}^\lambda),$$

$$k^\gamma_{\mu \nu} = \frac{\partial x^\gamma}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\nu} k_{\gamma \mu \nu} - \frac{\partial^2 x^\mu}{\partial x^\mu \partial x^\nu} \frac{\partial x^\lambda}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^\lambda},$$

we easily get

$$\mathfrak{u}_C = u^\lambda \partial_\lambda + (\partial_{\mu \nu} u^\lambda - k^\lambda_{\mu \nu} \partial_\mu u^\nu - k^\lambda_{\nu \mu} \partial_\nu u^\mu + k_{\nu \mu} \partial_\mu u^\lambda - k^\lambda_{\mu \nu} \partial_\nu u^\lambda) \partial_\lambda \mu \nu$$

(\partial_{\mu \nu} = \partial / \partial k^\lambda_{\mu \nu}). As one can easily check directly, (24) and (26) preserve the Lie bracket of vector fields, i.e. they are representations of $\mathcal{R}$-Lie algebras.

Using (16), we can write the corresponding vector field on the velocity phase space of the linear connections. Then taking the direct sum of this representation with (24),
we get the following representation of $R$-Lie algebras:

\[
\begin{align*}
\mathfrak{u} &= u^a \partial_a + u^{a\mu} \partial_{a\mu} + u^a \partial_u^a + u^{a\mu\nu} \partial^a_{\mu\nu} + u^{a\mu\nu\alpha} \partial^a_{\mu\nu\alpha}, \\
u^a &= g^{a\beta} \partial_a \nu^\beta + g^{a\beta} \partial_u^\beta, \\
u^{a\mu} &= \partial^a_{\mu\nu} u^a - k^{a\mu}_{\nu\alpha} \partial_u^\alpha + k^{a\mu}_{\nu\alpha} \partial_u^\alpha, \\
u^{a\mu\nu}_{\alpha\beta} &= \partial_a u^{a\mu\nu}_{\alpha\beta} - k^{a\mu\nu}_{\alpha\beta} \partial_u^\alpha + k^{a\mu\nu}_{\alpha\beta} \partial_u^\alpha, \quad \forall a, \beta, \mu, \nu, \alpha, \beta, \mu, \nu, \alpha.
\end{align*}
\]

(27)

Now let us consider a Lagrangian $L = L(x^A, g_{\mu\nu}, k^\alpha_{\beta\mu}, k_{\beta\mu, \mu}) \omega$. Then the condition for $L$ to be generally covariant is

\[
L \circ_\omega = 0
\]

(28)

for each vector field $u$ on the world manifold $X$.

Using the explicit expression given in (27), we see that (28) is equivalent to the following four conditions:

\[
\begin{align*}
\pi^{a\beta\mu} + \pi^{a\beta\mu} + \pi^{a\beta\mu} + \pi^{a\beta\mu} + \pi^{a\beta\mu} &= 0, \\
k^{a\beta}_{\mu\nu} \pi^{a\beta}_{\mu\nu} - k^{a\beta}_{\mu\nu} \pi^{a\beta}_{\mu\nu} - k^{a\beta}_{\mu\nu} \pi^{a\beta}_{\mu\nu} + \partial_{a\mu}^\gamma L + & + k^{a\beta}_{\mu\nu} \pi^{a\beta}_{\mu\nu} - k^{a\beta}_{\mu\nu} \pi^{a\beta}_{\mu\nu} = 0, \\
2g^{a\beta} \partial_a^\beta L + L k^{a\beta}_{\mu\nu} + k^{a\beta}_{\mu\nu} \partial_a^\beta L + & + k^{a\beta}_{\mu\nu} \partial_{a\mu\nu}^\beta L = 0, \\
\partial_a^\beta L &= 0,
\end{align*}
\]

(29-32)

where $k^{a\beta}_{\mu\nu} = \partial_{\mu\nu}^\beta$.

Remark 3.1. As in gauge theory [3], let us consider the curvature 2-form

\[
R = \frac{1}{2} R^a_{\beta\mu\nu} dx^a \wedge dx^\beta \otimes e_\eta, \quad e_\eta = dx^\beta \otimes \partial_a
\]

(34)

The 2-form $R$ can also be seen as a 2-form $\Omega$ in the following way:

\[
\Omega = [dk_{\beta\mu}^a \wedge dx^a + \frac{1}{2} (k_{\gamma\beta\mu}^a k_{\alpha\gamma}^a - k_{\gamma\beta\mu}^a k_{\alpha\gamma}^a) dx^a \otimes dx^\beta] \wedge e_\eta
\]

(35)

which looks like a generalized symplectic structure on the configuration space of the linear connections.
There is a canonical linear connection $D$ whose connection parameters are [8]

\[ \partial_{\gamma}x^\alpha \] $D_{\alpha}^\beta = 0 \]

\[ \partial_{\lambda}x^\alpha \] $D_{\alpha}^\beta = (k^\beta_{\alpha\lambda} \delta_{\alpha}^\gamma - k^\gamma_{\alpha\lambda} \delta_{\lambda}^\beta) \epsilon^\lambda \]

One can easily check that $\Omega$ is closed with respect to the exterior covariant derivative induced by $D$, i.e. $D\Omega = 0$. Of course, this is just the Bianchi identity.

Let $u = u^4 \partial_4$ be a vector field on $X$. Then, denoting by $\nabla$ the covariant derivative associated with a linear connection $k^\alpha_{\beta\lambda}$, we have

\[ \nabla u = (\partial_{\lambda} u^\alpha - k^\alpha_{\beta\lambda} u^\beta) \epsilon^\lambda \]

Using the torsion

\[ T = \frac{1}{2} (k^\alpha_{\beta\lambda} - k^\alpha_{\lambda\beta}) \, dx^\beta \wedge dx^\lambda \]

we can modify $\nabla$ by adding $T$, thereby getting the new connection

\[ \tilde{\nabla} u = \nabla u + u \epsilon^4 \]

Now applying the canonical covariant derivatives (36) and (37) to $\tilde{\nabla} u$, we find

\[ D(\tilde{\nabla} u) = \{ -u^a \epsilon^b \epsilon^a_{\beta\lambda} + [u^a_{\lambda\beta} + (k^\gamma_{\beta\lambda} k^\alpha_{\gamma\mu} - k^\gamma_{\gamma\mu} k^\alpha_{\lambda\beta}) u^\lambda] \} \epsilon^\gamma \]

where $u^a_{\lambda\beta}$ is given by (27).

It follows immediately that the vector field (26) is uniquely determined by the condition

\[ u_c \epsilon^4 \] $D(\tilde{\nabla} u) = 0 \]

where $\Omega$ is given by (35). Thus the vector field $u_c$ is the Hamiltonian vector field associated with $u$.

3.2. Dependence through the curvature. Let us consider a Lagrangian density whose dependence on the linear connections on $X$ is only through their curvatures, i.e.

\[ \mathcal{L} = \mathcal{F} \circ R \]

where $R$ is given by (34). It is easily seen that the condition (43) is equivalent to the two following ones [3]:

\[ \pi^a_{\beta\mu} + \pi^a_{\mu\beta} = 0 \]

\[ \partial_{\gamma}x^\lambda + k^\gamma_{\gamma\mu} \pi^a_{\mu\lambda} - k^\beta_{\alpha\mu} \pi^a_{\gamma\lambda} = 0 \]

Of course, the conditions (44) and (45) imply the conditions (29) and (30), respectively. The other two conditions (31) and (32) are just equivalent to $\mathcal{F}$ being invariant under the infinitesimal transformations of the world manifold $X$. 
4. - Currents and superpotentials

Let us consider a generally covariant Lagrangian density $\mathcal{L}$ as in (43). Recalling (11), we see that the field operator for the metric is simply

\[(46) \quad \mathcal{E}_{\alpha\beta} = \partial_{\alpha\beta} L,\]

while that for the connection it is

\[(47) \quad \mathcal{E}_{\alpha}^{\beta\lambda} = \partial_{\alpha}^{\beta\lambda} L - d_{\mu} \tau_{\alpha}^{\beta\lambda}.\]

From (18) we get the current

\[(48) \quad J^{\lambda} = u^{\lambda} L - u^{\mu} k^{\alpha}_{\beta\nu,\mu} \pi^{\beta\nu\lambda} + (k^{\gamma}_{\beta\nu} \partial_{\gamma} u^{\mu} - k^{\alpha}_{\beta\nu} \partial_{\alpha} u^{\mu} - k^{2}_{\beta\nu} u^{\mu}) \pi^{\beta\nu\lambda},\]

where we have used (26).

We have the following lemma.

**Lemma 4.1.** Using only the condition (31), the current (48) can be written as

\[(49) \quad J^{\lambda} = -(2g^{\alpha\beta} \mathcal{E}_{\alpha\beta} + k^{\lambda}_{\beta\nu} \mathcal{E}_{\alpha}^{\beta\nu} - k^{\alpha}_{\beta\nu} \mathcal{E}_{\beta}^{\lambda\nu} - k^{\beta}_{\nu} \mathcal{E}_{\alpha}^{\nu}\lambda - d_{\mu}(k^{\lambda}_{\beta\nu} \tau_{\alpha}^{\beta\nu} - k^{\beta}_{\nu} \tau_{\alpha}^{\lambda\nu}) u^{\alpha} +
\]

\[+ (k^{\gamma}_{\beta\nu} \partial_{\gamma} u^{\mu} - k^{\alpha}_{\beta\nu} \partial_{\alpha} u^{\mu} + d_{\mu}(\pi^{\beta\lambda}_{\alpha} \partial_{\beta} u^{\alpha} +
\]

\[+ (k^{\gamma}_{\beta\nu} \partial_{\gamma} u^{\mu} - k^{\alpha}_{\beta\nu} \partial_{\alpha} u^{\mu} + d_{\mu}(\tau^{\beta\lambda}_{\alpha} \partial_{\beta} u^{\alpha}).\]

**Proof.** Using (46) and (47), the condition (31) can be rewritten as follows:

\[(50) \quad 2g^{\alpha\beta} \mathcal{E}_{\alpha\beta} + L \delta^{\lambda}_{\alpha} - k^{\beta}_{\nu} \mathcal{E}_{\alpha}^{\beta\nu} + k^{\lambda}_{\beta\nu} \mathcal{E}_{\alpha}^{\beta\nu} + k^{\lambda}_{\beta\nu} d_{\mu} \mathcal{E}_{\alpha}^{\beta\nu} + k^{\lambda}_{\beta\nu} \mathcal{E}_{\alpha}^{\beta\nu} - k^{\lambda}_{\nu} \mathcal{E}_{\alpha}^{\lambda\nu} - k^{\lambda}_{\nu} \mathcal{E}_{\alpha}^{\nu}\lambda - k^{\beta}_{\nu} \mathcal{E}_{\alpha}^{\nu}\lambda - k^{\beta}_{\nu} \mathcal{E}_{\alpha}^{\nu}\lambda = 0,
\]

or, equivalently,

\[(51) \quad L \delta^{\lambda}_{\alpha} - k^{\beta}_{\nu} \mathcal{E}_{\alpha}^{\beta\nu} = -(2g^{\alpha\beta} \mathcal{E}_{\alpha\beta} + k^{\lambda}_{\beta\nu} \mathcal{E}_{\alpha}^{\beta\nu} - k^{\beta}_{\nu} \mathcal{E}_{\alpha}^{\nu}\lambda - k^{\beta}_{\nu} \mathcal{E}_{\alpha}^{\nu}\lambda - d_{\mu}(k^{\lambda}_{\beta\nu} \tau_{\alpha}^{\beta\nu} - k^{\beta}_{\nu} \tau_{\alpha}^{\lambda\nu}).\]

Now the result (49) follows immediately by substituting (51) into the expression (48) of the current $J^{\lambda}$. ■

Let us consider the last three rows in (49) separately. The first row can be rewritten as follows:

\[(52) \quad -d_{\mu}(k^{\lambda}_{\beta\nu} \tau_{\alpha}^{\beta\nu} - k^{\beta}_{\nu} \tau_{\alpha}^{\lambda\nu}) u^{\alpha} = d_{\mu}(\mathcal{E}_{\alpha}^{\lambda\mu} L) u^{\alpha} =
\]

\[= d_{\mu}(\mathcal{E}_{\alpha}^{\lambda\mu} + d_{\mu} \tau_{\alpha}^{\lambda\nu} L) u^{\alpha} = d_{\mu}(\mathcal{E}_{\alpha}^{\lambda\mu}) u^{\alpha},\]
where we have used (44) and (45). The second row leads to
\[ \frac{d}{d\mu}(k^{\beta}_{\alpha \gamma} \pi^{\nu \mu}) u^\alpha - k^{\alpha \gamma} \pi^{\nu \mu} \frac{\partial}{\partial \mu} u^\nu + d_{\mu}(\pi^{\nu \mu} \pi^{\nu \mu}) u^\alpha = \]
\[ = - d_{\mu}(\pi^{\nu \mu} \pi^{\nu \mu} \frac{\partial}{\partial \mu} u^\nu) = d_{\mu}(\pi^{\nu \mu} \pi^{\nu \mu} \frac{\partial}{\partial \mu} u^\nu), \]
where we have used (44). The expression of the covariant derivative \( \nabla_{\mu} u^\alpha \) is that in (38).

Finally, the third row yields
\[ (k^{\gamma}_{\rho \nu} \frac{\partial}{\partial \nu} u^\alpha - k^{\gamma}_{\rho \nu} \frac{\partial}{\partial \nu} u^\nu) \pi^{\beta \alpha} - d_{\mu}(\pi^{\beta \alpha} \frac{\partial}{\partial \mu} u^\alpha) = \]
\[ = (k^{\gamma}_{\rho \nu} \pi^{\beta \alpha} - k^{\gamma}_{\rho \nu} \pi^{\beta \alpha}) \frac{\partial}{\partial \nu} u^\alpha - d_{\mu}(\pi^{\beta \alpha} \frac{\partial}{\partial \mu} u^\alpha) = \]
\[ = (\frac{\partial}{\partial \nu} + d_{\mu}(\pi^{\beta \alpha} \frac{\partial}{\partial \mu}) \frac{\partial}{\partial \mu} u^\alpha = - \pi^{\beta \alpha} \frac{\partial}{\partial \mu} \frac{\partial}{\partial \mu} u^\alpha, \]
where we have used (44) and (45).

Collecting these results together, we get the following proposition.

Proposition 4.2. Let \( L \) be a generally covariant Lagrangian density as in (43). Then, on the solutions of the field equations associated with (46) and (47), the corresponding superpotential is
\[ U^{\lambda \mu} = \pi^{\beta \alpha} \{ [\nabla_{\mu} u^\alpha + (k^{\alpha \gamma} - k^{\alpha \gamma}) u^\gamma] \pi^{\beta \alpha} \} \nabla_{\beta} u^\alpha, \]
where \( \nabla \) has been defined in (40).

Of course, the superpotential (55) is defined up to a closed \((m - 2)\)-form. Note that the covariant derivative \( \nabla u \) appearing in the superpotential (55) is just the same which determines the Hamiltonian lift \( u_C \) in (42).

Remark 4.3. From (49), using (52), (53) and (54), we see that the current \( J^\lambda \) can be written in the following way:
\[ J^\lambda = d_{\mu}(\pi^{\beta \alpha} \nabla_{\mu} u^\alpha) - 2g^{\beta \alpha} E_{\alpha \beta} - \pi^{\beta \alpha} \nabla_{\beta} u^\alpha + \]
\[ + (d_{\mu} E_{\alpha}^{\lambda \mu} - k^{\lambda}_{\beta \mu} E_{\alpha}^{\beta \mu} + k^{\beta \alpha} E_{\alpha}^{\lambda \mu}) u^\alpha. \]

The intrinsic character of each term of the first row in (56) is obvious. It may be interesting to see directly that also the second row in (56) is intrinsic. Indeed, we have
\[ (d_{\mu} E_{\alpha}^{\lambda \mu} - k^{\lambda}_{\beta \mu} E_{\alpha}^{\beta \mu} + k^{\beta \alpha} E_{\alpha}^{\lambda \mu}) u^\alpha = \]
\[ = d_{\mu}(E_{\alpha}^{\lambda \mu} u^\alpha) - k^{\lambda}_{\beta \mu} E_{\alpha}^{\beta \mu} u^\alpha - E_{\alpha}^{\lambda \mu} \nabla_{\mu} u^\alpha = \nabla_{\mu}(E_{\alpha}^{\lambda \mu} u^\alpha) - E_{\alpha}^{\lambda \mu} \nabla_{\mu} u^\alpha, \]
where \( \nabla_{\mu}(E_{\alpha}^{\lambda \mu} u^\alpha) \) denotes the exterior covariant derivative of the tangent valued \((m - 1)\)-form \( E_{\alpha}^{\lambda \mu} u^\alpha \). Thus the covariant character of the current (56) is manifest.

***

This work was supported by GNFM-CNR, by the National Research Project 40% "Geometria e Fisica" of the MURST, and by the University of Camerino.
REFERENCES